

# Connections and dynamical trajectories in generalised Newton-Cartan gravity II.

## An ambient perspective

Xavier Bekaert & Kevin Morand

Laboratoire de Mathématiques et Physique Théorique

Unité Mixte de Recherche 7350 du CNRS

Fédération de Recherche 2964 Denis Poisson

Université François Rabelais, Parc de Grandmont

37200 Tours, France

Xavier.Bekaert@lmpt.univ-tours.fr

Kevin.Morand@lmpt.univ-tours.fr

### Abstract

Connections compatible with degenerate metric structures are known to possess peculiar features: on the one hand, the compatibility conditions involve restrictions on the torsion; on the other hand, torsionfree compatible connections are not unique, the arbitrariness being encoded in a tensor field whose type depends on the metric structure. Nonrelativistic structures typically fall under this scheme, the paradigmatic example being a contravariant degenerate metric whose kernel is spanned by a one-form. Torsionfree compatible (*i.e.* Galilean) connections are characterised by the gift of a two-form (the force field). Whenever the two-form is closed, the connection is said Newtonian. Such a nonrelativistic spacetime is known to admit an ambient description as the orbit space of a gravitational wave with parallel rays. The leaves of the null foliation are endowed with a nonrelativistic structure dual to the Newtonian one, dubbed Carrollian spacetime. We propose a generalisation of this unifying framework by introducing a new ambient metric structure of which we study the geometry. We characterise the space of (torsional) connections preserving such a metric structure which is shown to project to (resp. embed) the most general class of (torsional) Galilean (resp. Carrollian) connections. In particular, we show that torsionfree Galilean connections can be obtained by projection of a torsional connection compatible with an ambient Lorentzian metric, the force field arising from the light-like part of the ambient torsion.

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# 1 Introduction

In contradistinction with the familiar relativistic case, the geometry of nonrelativistic spacetimes is characterised by *degenerate* metric structures. The princeps example of nonrelativistic spacetimes originates from Newtonian mechanics, in which time and space are assumed to be separate entities, each one being imparted with an absolute status. Both of these features can be nicely encoded in a geometric fashion [1, 2] by identifying the nonrelativistic spacetime as a manifold  $\bar{\mathcal{M}}$  endowed with a metric structure  $(\bar{h}^{\mu\nu}, \bar{\psi}_\mu)$ , where  $\bar{h}^{\mu\nu}$  stands for a twice-contravariant degenerate metric whose radical is spanned by a nowhere-vanishing 1-form  $\bar{\psi}_\mu$  (hence  $\bar{h}^{\mu\nu}\bar{\psi}_\mu = 0$ ). Nowadays, this construction is often referred to as *Newton-Cartan geometry*<sup>1</sup>. The 1-form  $\bar{\psi}$  plays here the role of an *absolute clock* while the cometric  $\bar{h}$  provides  $\bar{\mathcal{M}}$  with a notion of spatial distance and will hence be referred to as a collection of *absolute rulers*. An equivalent definition of absolute rulers can be given in the form of a positive-definite metric  $\bar{\gamma}$  acting on the kernel of  $\bar{\psi}$ . When causality is assumed (*i.e.* when the absolute clock satisfies the Frobenius integrability condition  $\bar{\psi}_{[\mu}\partial_\lambda\bar{\psi}_{\nu]} = 0$ ), the kernel of  $\bar{\psi}$  defines an integrable distribution whose integral submanifolds foliate the spacetime  $\bar{\mathcal{M}}$ . Each of these integral hypersurfaces can then be thought of as the *absolute space* at a given time on which a notion of distance is provided by the collection of absolute rulers  $\bar{\gamma}$ . The “metric” structure of Newtonian gravity is thus embodied in a triplet  $(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$ , referred to as a *Leibnizian structure* in [4, 5].

Strictly speaking, metric structures (degenerate or not) do not provide a notion of parallelism. The latter can be implemented in the guise of a compatible connection. However, connections compatible with degenerate metric structures are known [6] to differ from the usual, nondegenerate case by at least two related aspects: on the one hand, the compatibility conditions involve restrictions on the torsion so that not all torsion tensors are admissible; on the other hand, only a subset of metric structures admit a torsionfree compatible connection. Moreover, even when the existence of such a connection is ensured, its uniqueness is not and two different torsionfree compatible connections differ by a tensor field whose type depends on the metric structure. As for Leibnizian structures, the compatibility condition with the absolute clock  $(\bar{\nabla}_\mu\bar{\psi}_\nu = 0)$  enforces the following constraint on the torsion tensor:

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<sup>1</sup>For textbooks including reviews on Newton-Cartan geometry and gravity, see *e.g.* [3].

$\bar{\psi}_\lambda \bar{\Gamma}_{[\mu\nu]}^\lambda = \partial_{[\mu} \bar{\psi}_{\nu]}$ . Consequently, only Leibnizian structures with *closed* absolute clock admit torsionfree compatible connections (this subset was dubbed *Augustinian* structures in [5]). Consequently, when dealing with nonrelativistic metric structures with non-closed absolute clocks ( $d\bar{\psi} \neq 0$ ), torsionfreeness and compatibility become mutually exclusive features. Furthermore, the previous constraint fixes the “timelike” part of the torsion tensor so that the gift of the latter is not sufficient to uniquely define a compatible connection and must be supplemented by a 2-form  $\bar{F}_{\mu\nu}$ .

When the absolute clock is closed, torsionfree compatible connections are allowed and are then completely characterised by the gift of a 2-form  $\bar{F}$ . The appearance of such a 2-form in the nonrelativistic context is in fact quite natural if one acknowledges the fact that in Newtonian mechanics, spacetime is a mere “container” whose structure is not rich enough to prescribe the motion of particles. When the external forces are of the Lorentz type (*i.e.*  $\ddot{x}^\mu = \bar{h}^{\mu\nu} \bar{F}_{\nu\rho} \dot{x}^\rho$ ), the dynamics of a single particle is specified by a force-field  $\bar{F}$  (interpreted as the Faraday tensor in the electromagnetic case). Of particular physical interest is the subcase of force fields deriving from a potential 1-form, *i.e.* when  $\bar{F}$  is closed so that locally there exists a 1-form  $\bar{A}$  such that  $\bar{F} = d\bar{A}$ . Remarkably, the condition that the force-field 2-form associated to a given Galilean connection is closed can be translated in an algebraic condition on the curvature of the connection, the so-called Duval-Künzle condition [7, 8]. Galilean connections satisfying this extra closedness condition are traditionnally called *Newtonian* connections. In such case, the equations of motion prescribing the dynamics of a particle can be derived via a variational principle from a Lagrangian built in terms of the Leibnizian structure and the potential 1-form. As shown in [9, 5], such a Lagrangian defines a (possibly nondegenerate) *Lagrangian metric* on  $\bar{\mathcal{M}}$  which reduces to the absolute collections of rulers  $\bar{\gamma}$  on the absolute spaces and furthermore constitutes the necessary and sufficient structure needed to supplement a Leibnizian structure in order to define a unique Newtonian connection.

The following diagram sums up the interrelations between the different *dramatis personæ* constituting the kinematical content of Newton-Cartan geometry, in complete analogy with the relativistic case:

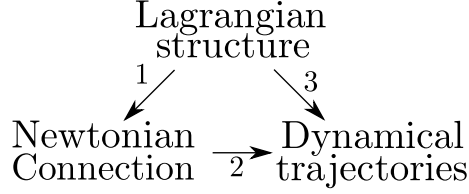


Figure 1: Kinematical content of Newton-Cartan geometry

Besides its purely mathematical interest, Newton-Cartan geometry has recently known a revival of interest triggered firstly by condensed-matter applications [10, 11] following the seminal work by Son [12] (*cf.* also [13] for an early inspiring work on superfluids). These structures have also made novel appearances in the active fields of Lifshitz and Schrödinger holography [14, 15] and in the context of Hořava-Lifshitz gravity [16]. Most of these recent works focus (with the exception of the work [17] which consider the general case) on a special class of torsional Galilean connection dubbed Torsional Newton-Cartan (TNC) geometries .

### Newtonian structures embedded inside Bargmannian structures

Although Newtonian structures have been advocated to live on their own, new light can be shed on such structures by embedding them inside relativistic ones, thus providing naturality to seemingly peculiar nonrelativistic structures and properties by importing them from usual relativistic ones. The origin of this perspective on nonrelativistic physics can be traced back to an early work of Eisenhart [18] establishing that the dynamical trajectories of a holonomic mechanical system with  $d$  degrees of freedom can be put in correspondence with the affine geodesics of a specific  $d+2$  dimensional relativistic spacetime. Thus, nonrelativistic dynamical trajectories can always be “lifted” to geodesics (hence the denomination of “Eisenhart lift”) and conversely, any relativistic geodesic can be projected onto a nonrelativistic dynamical trajectory. The class of relativistic spacetimes allowing the Eisenhart lift can be characterised by the existence of a light-like vector field which is parallel with respect to the Levi-Civita connection. This class of metrics had previously been considered

in [19] in a different context and later received the interpretation of gravitational waves with parallel rays [20], where the rays are the integral curves of the light-like and parallel vector field (dubbed *wave vector field* in the following). Such spacetimes will be referred to as *Bargmann-Eisenhart waves*, cf. [21].

This bridge between nonrelativistic physics and relativistic spacetimes has been independently rediscovered afterwards by Duval and collaborators [22, 20] who generalised the “ambient” approach of Eisenhart in order to provide an account of the different levels of the kinematical and dynamical contents of Newtonian gravity as embedded inside Bargmann-Eisenhart waves.

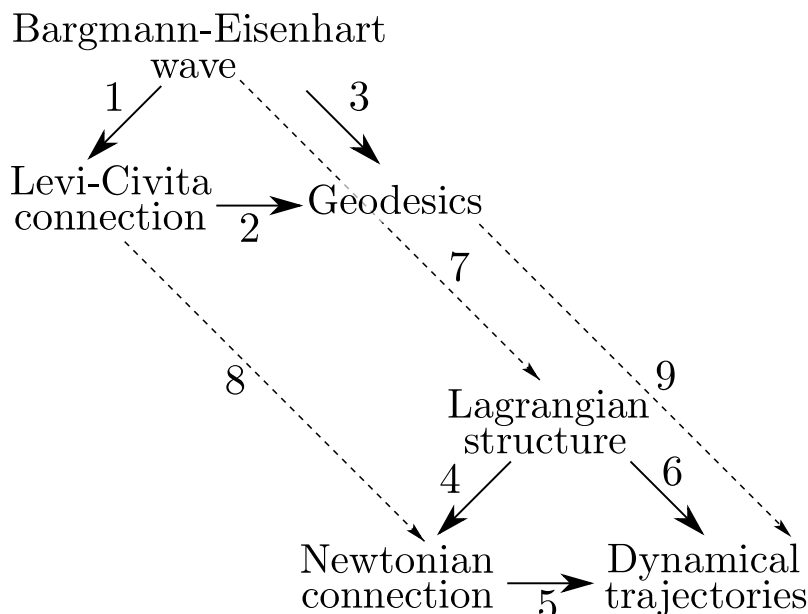


Figure 2: Newtonian structures embedded inside Bargmannian structures

The main idea underlying the ambient approach to nonrelativistic physics consists in performing a dimensional reduction of a Bargmann-Eisenhart wave along the null rays, thus differing from the usual Kaluza-Klein framework in which the reduction typically occurs along a spacelike direction (or even from the timelike dimensional reduction for stationary spacetimes). The quotient manifold  $\bar{\mathcal{M}}$  obtained by dimensional reduction of a Bargmann-Eisenhart wave along the null rays thus inherits a structure of Newtonian spacetime, which can be appreciated at the different levels of structure, as depicted on Figure 2. Nonrelativistic structures thus appears as “shadows” of relativistic ones. In this aspect, the ambient formalism is reminiscent of Plato’s allegory of the Cave [23] (cf. [24, 21] for this analogy) which depicts material phenomena as mere shadows of pure Forms. Explicitly, the metric of a Bargmann-

Eisenhart wave projects onto the quotient manifold (dubbed *Platonic screen* in the following) as the Lagrangian (Arrow 7) while the Levi-Civita connection associated to the relativistic metric structure defines a nonrelativistic Newtonian connection (Arrow 8) related to the corresponding Lagrangian structure. The Eisenhart lift, understood as a correspondence between nonrelativistic dynamical trajectories and relativistic geodesics, is symbolised by Arrow 9.

Since its introduction in [22, 20], the ambient formalism has been successfully used to approach a wide range of nonrelativistic problems, such as Chern-Simons electrodynamics [25], fluid dynamics [26], Newton-Hooke cosmology [27], Schrödinger symmetry [28], Kohn's theorem [29], hidden symmetries [30], *etc.*

### Nonrelativistic theories of gravitation and kinematical algebras

The key to the ambient approach followed in [22, 20] lies in the interplay between algebraic and geometric structures, the origin of which, in a nonrelativistic context, can be traced back to the seminal work of Künzle [7] in which Leibnizian structures were obtained as  $G$ -structures for the homogeneous Galilei group. Similarly, Bargmann-Eisenhart waves are defined as  $G$ -structures for the homogeneous Bargmann group<sup>2</sup> [22, 20]. It is indeed hard to overstate the relevance of the Bargmann group (the central extension of the Galilei group [31]) when the geometrisation of nonrelativistic physics is concerned, both in the intrinsic or ambient fashion. From an intrinsic viewpoint, the Bargmann group has proved to be very useful in order to reformulate the Duval-Künzle condition and thus to deal with Newtonian connections (*cf.* [8] for an approach in terms of affine connections, [32, 17] in the context of gauging procedures and [33] in the formalism of Cartan geometries). From an ambient standpoint, these approaches rely crucially on the group-theoretical avatar of the ambient formalism, namely the embedding of nonrelativistic symmetry groups (*e.g.* the Bargmann [31] and Schrödinger [34] groups) inside their relativistic homologues (Poincaré [35] and conformal [36] groups, respectively). These nonrelativistic groups can thus be obtained from their relativistic counterparts by the group-theoretical analogue of a light-like dimensional reduction, namely as subgroups preserving a light-like direction.

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<sup>2</sup>By homogeneous Bargmann group we intend the homogeneous Galilei group in a  $d + 2$  dimensional representation inherited from that of the Bargmann group, *cf.* [22].

Besides its importance for Newton-Cartan geometry, group-theoretical input has been used in order to describe other nonrelativistic structures. In a seminal paper [37], Bacry and Lévy-Leblond displayed a classification of the so-called “kinematical” algebras, namely algebras which encode the infinitesimal symmetries of a free particle. Their classification distinguishes between “relativistic” (Poincaré, (anti) de Sitter) and nonrelativistic algebras (Galilei, Newton-Hooke, Carroll). As the Galilei group can be thought of as a nonrelativistic avatar of the Poincaré group, Newton-Hooke algebras can be seen as nonrelativistic equivalent of the (anti) de Sitter algebras. From an ambient standpoint, the Newton-Hooke spacetime can be obtained as a light-like dimensional reduction from an Hpp-wave, whose isometry group coincides precisely with the central extension of the Newton-Hooke group [27]. Recently, the Carroll group, introduced in [38], has known a new topicality following the works [39, 40, 41] (*cf.* also [42, 43]) unravelling the relation with the Bondi-Metzner-Sachs (BMS) group which can be characterised as conformal extension of the Carroll group. Furthermore, the work [39] displayed an interesting duality between the Galilei and Carroll groups, both groups can be obtained from the Bargmann group, by projection and embedding respectively. This duality has a natural geometric counterpart: on the one hand, Newtonian manifolds can be obtained from a Bargmann-Eisenhart wave by projection on the Platonic screen; on the other hand, the leaves orthogonal to the wave vector field of a Bargmann-Eisenhart wave are endowed with a Carrollian structure (*i.e.* a nowhere-vanishing vector field  $\tilde{\xi}$  spanning the radical of a twice-covariant degenerate metric  $\tilde{\gamma}$ ) inherited from the Bargmann-Eisenhart structure  $(g, \xi)$  while the Levi-Civita connection associated to the ambient metric  $g$  induces on each leaf a connection compatible with the Carrollian structure (and is hence referred to as a *Carrollian* connection).

We note that the nonrelativistic structures induced by a Bargmann-Eisenhart wave are not the most general one can consider. On the one hand, the metric structure obtained by projection on the Platonic screen is Augustinian (*i.e.*  $d\bar{\psi} = 0$ ) and the projected connection is Newtonian. On the other hand, the extrinsic curvature of the leaves foliating a Bargmann-Eisenhart wave vanishes, so that the induced Carrollian structure is necessarily invariant. Furthermore, the twice-covariant tensor encoding the arbitrariness in the choice of a Carrollian connection must be equal to the “transverse extrinsic curvature” (in the terminology of [44]).



## 1.1 Outline of the paper

In the present work we propose a generalisation of the ambient setup of [22, 20] which will allow to embed the most general nonrelativistic structure in an ambient manifold. This is achieved via the introduction of a new ambient metric structure (dubbed *ambient Leibnizian structure* in the following) of which we study the geometry. Explicitly, an ambient Leibnizian structure is defined as a quadruplet  $(\mathcal{M}, \xi, \psi, \gamma)$  where  $\xi$  is a nowhere-vanishing vector field on the manifold  $\mathcal{M}$ , the 1-form  $\psi$  is an absolute clock annihilating  $\xi$  and the twice-covariant “metric”  $\gamma$  is defined on the kernel of  $\psi$  and its radical is spanned by  $\xi$ . This new metric structure, despite being neither Leibnizian nor Carrollian, provides a unifying ambient framework allowing to embed the most general classes of both Leibnizian and Carrollian structures. We now give an outline of the paper and summarise our main results:

Section 2.1 consists in a review of standard material regarding principal  $\mathbb{R}$ -bundles in order to fix some terminology. The term *Platonic screen* is introduced to designate the orbit space  $\bar{\mathcal{M}}$  (*i.e.* the base space) of such a principal  $\mathbb{R}$ -bundle  $\mathcal{M}$ . We then review various isomorphisms between spaces of ambient and base structures.

In Section 2.2, we introduce the notion of ambient Leibnizian structure. The algebra (coined the Leibniz algebra  $\mathfrak{leib}(d+2)$ ) of infinitesimal automorphisms preserving a flat ambient Leibnizian structure is shown to be a semidirect sum of the Abelian ideal  $\mathbb{R}^{d+2}$  of infinitesimal translations with the (inhomogeneous) Carroll algebra  $\mathfrak{carr}(d+1)$  as “homogeneous” subalgebra (*i.e.* fixing a point), so that ambient Leibnizian structures can be thought of as  $G$ -structures for the *inhomogeneous* Carroll group. The necessary and sufficient conditions for an ambient Leibnizian structure to be projectable on the Platonic screen are given. Any ambient Leibnizian structure can be promoted to a Lorentzian manifold by adding a suitable ingredient that we identify. We characterise the affine space of such Lorentzian metrics and determine the associated model vector space. The subalgebra of infinitesimal automorphisms preserving such a Lorentzian metric is shown to be isomorphic to the Bargmann algebra  $\mathfrak{bar}(d+2) \subset \mathfrak{leib}(d+2)$ .

Section 2.3 deals with the class of gravitational waves, *i.e.* Lorentzian spacetimes admitting a null and hypersurface-orthogonal vector field. The hypersurface-orthogonality condition ensures that gravitational waves possess an ambient Leib-

nizian structure whose absolute clock satisfies the Frobenius integrability condition ( $\psi \wedge d\psi = 0$ ) The ambient spacetime  $\mathcal{M}$  is then foliated by a family of codimension-one null hypersurfaces (dubbed *wavefront worldvolumes*), endowed with a Carrollian structure inherited from the ambient metric structure. In this context, Kundt waves are characterised as the most general class of gravitational waves that project onto a nonrelativistic causal structure (*i.e.* with  $\bar{\psi} \wedge d\bar{\psi} = 0$ ).

Section 3 is dedicated to the study of connections compatible with an ambient Leibnizian structure, dubbed *ambient Galilean* connections. Section 3.1 deals with the torsionfree case. The equivalence problem (*i.e.* the search for the necessary data supplementing the metric structure in order to unambiguously fix the compatible connection) is considered. The arbitrariness in the choice of an ambient Galilean connection is then shown to be encoded in a couple constituted by a 2-form and a symmetric covariant rank-2 tensor, corresponding to the respective arbitrariness in the choice of a torsionfree connection respectively Galilean or Carrollian. Furthermore, this solution to the equivalence problem allows us to construct two surjective maps from the space of nonrelativistic torsionfree ambient Galilean connections to the space of torsionfree Galilean and Carrollian connections, respectively. In other words, any torsionfree Galilean (resp. Carrollian) manifold can be obtained by projection of (resp. embedding inside) an ambient Galilean manifold.

These results are generalised to the torsional case in Section 3.2. We characterise the affine space of torsional ambient Galilean connections, of which the expression in components is given by eq.(3.49). We introduce a collection of privileged origins (dubbed *torsional special ambient connections* in the following) in this infinite-dimensional affine space and construct a surjective map from the space of torsional ambient Galilean manifolds to the space of nonrelativistic torsional Galilean manifolds.

In Section 3.3, we identify the subclass of ambient Galilean connections compatible with a Lorentzian metric extending the ambient Leibnizian structure. In the torsionfree case, we recover the results of [22, 20] stating that the compatibility condition forces the (previously arbitrary) 2-form to be closed, so that the projected torsionfree connection is necessarily Newtonian. Moreover, we identify the corresponding restrictions on the Carrollian structure: the symmetric twice-covariant tensor must coincide with the transverse extrinsic curvature of the considered wave-

front worldvolume. This class of Lorentzian manifolds is then shown to possess the sufficient arbitrariness to embed the whole class of nonrelativistic Galilean manifolds. Two particular cases are discussed: on the one hand, *torsionfree* Galilean connections (with generic force field  $\bar{F}$ ) are shown to arise as projection of *torsional* (Lorentzian metric compatible) connections where the force field is inherited from the “light-like part” (defined below) of the ambient torsion. On the other hand, we discuss the embedding of TNC geometries inside ambient torsional Lorentzian manifolds and display the explicit form of the torsion tensor.

Two appendices conclude this work. Appendix A is devoted to Carrollian manifolds. We first review the construction of [45, 39] related to Carrollian metric structures and then investigate Carrollian connections. We characterise the affine spaces of such connections in the torsionfree and torsional case and display component expressions for the most general Carrollian connection. Appendix B collects some technical proofs.

## 1.2 Notations

We will mostly follow the notations of [5].

Let  $V$  be a vector space and  $v, w \in V$  two vectors. We will denote by  $v \vee w = \frac{1}{2}(v \otimes w + w \otimes v)$  (respectively  $v \wedge w = \frac{1}{2}(v \otimes w - w \otimes v)$ ) the (anti)symmetric product, and similarly for higher products. The (anti)symmetrisation of indices is performed with weight one and is denoted by round (respectively, square) brackets, *e.g.*  $\Phi_{(\mu\nu)} \equiv \frac{1}{2}(\Phi_{\mu\nu} + \Phi_{\nu\mu})$  and  $\Phi_{[\mu\nu]} \equiv \frac{1}{2}(\Phi_{\mu\nu} - \Phi_{\nu\mu})$ .

Let  $\mathcal{V}$  be a vector bundle over  $\mathcal{M}$  with typical fibre the vector space  $V$ . By  $\Gamma(\mathcal{V})$ , we will denote the space of its sections, *i.e.* globally defined  $V$ -valued fields on  $\mathcal{M}$ . In contrast with [5], the manifold  $\mathcal{M}$  will stand for the ambient relativistic spacetime manifold of dimension  $d + 2$ . Barred quantities will most often denote nonrelativistic objects defined on the  $(d + 1)$ -dimensional nonrelativistic spacetime  $\bar{\mathcal{M}}$ .

## 2 Ambient spacetimes as principal bundles

This section introduces the basic ambient setup, *i.e.* the ambient manifold and its various metric structures prior to the introduction of a compatible connection. In Section 2.1, we recall the geometric definition of an ambient manifold as a principal  $\mathbb{R}$ -bundle  $\mathcal{M}$  with fundamental vector field  $\xi$  and introduce some terminology (proposed in [21]) for the corresponding base space  $\bar{\mathcal{M}}$  and sections. More importantly, we discuss the necessary and sufficient conditions for (covariant and contravariant) tensor fields on  $\mathcal{M}$  to be projectable on  $\bar{\mathcal{M}}$ . These criteria will be repeatedly used in the paper. In Section 2.2 we introduce various notions of ambient (Leibnizian, Lorentzian, Lagrangian) metric structures while in Section 2.3 we show that Kundt waves provide the most general ambient Lorentzian spacetime  $\mathcal{M}$  such that the base manifold  $\bar{\mathcal{M}}$  is a nonrelativistic spacetime foliated by absolute spaces.

### 2.1 Ambient geometry

Let  $\mathcal{M}$  be a manifold endowed with a complete and nowhere-vanishing vector field  $\xi \in \Gamma(T\mathcal{M})$ . The integral curve  $\gamma_x : \mathbb{R} \rightarrow \mathcal{M} : \lambda \mapsto \gamma_x(\lambda)$  of the vector field  $\xi$ , passing through the point  $x \in \mathcal{M}$ , is the unique solution to the differential equation

$$\frac{d\gamma_x(\lambda)}{d\lambda} = \xi|_{\gamma_x(\lambda)} \quad (2.1)$$

with  $\lambda \in \mathbb{R}$  and initial condition  $\gamma_x(0) = x$ . Since  $\xi$  is assumed to be complete, its integral curves exist for all values of the parameter  $\lambda$  and, consequently, the flow

$$\varphi_\xi : \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M} : (x, \lambda) \mapsto \gamma_x(\lambda) \quad (2.2)$$

is global thus inducing a well-defined right-action of the additive Lie group  $\mathbb{R}$  on  $\mathcal{M}$ . Since  $\xi$  is also assumed to be nowhere-vanishing, this action is *free*. The integral curve  $I_x = \{\gamma_x(\lambda) \in \mathcal{M} \mid \lambda \in \mathbb{R}\}$  through  $x \in \mathcal{M}$  is thus the orbit of  $x$  under the  $\mathbb{R}$ -action  $\varphi_\xi$ .

**Definition 2.1** (Platonic screen [21]). *Let  $\mathcal{M}$  be a manifold endowed with a complete and nowhere-vanishing vector field  $\xi \in \Gamma(T\mathcal{M})$ . The Platonic screen of  $\mathcal{M}$  is the*

orbit space  $\bar{\mathcal{M}}$  of the action  $\varphi_\xi$ , i.e. the set  $\bar{\mathcal{M}} \equiv \mathcal{M}/\mathbb{R} = \{I_x | x \in \mathcal{M}\}$  of all integral curves on  $\mathcal{M}$ .

The quotient manifold Theorem (cf. e.g. Theorem 21.10 in [46]) ensures that if the  $\mathbb{R}$ -action  $\varphi_\xi$  on  $\mathcal{M}$  is also proper (which we will always assume), then the Platonic screen is a manifold. More precisely, the projection map onto orbits of  $\varphi_\xi$ , denoted  $\pi : \mathcal{M} \rightarrow \bar{\mathcal{M}} : x \mapsto I_x$  is a submersion and therefore defines the principal fiber bundle:

$$\mathbb{R} \hookrightarrow \mathcal{M} \xrightarrow{\pi} \bar{\mathcal{M}} \quad (2.3)$$

whose fibers are the integral curves  $I_x$  and whose fundamental vector field is  $\xi$ . The vertical subspace  $V_x \subset T_x\mathcal{M}$  at a point  $x \in \mathcal{M}$  is therefore spanned by  $\xi_x$ , i.e.  $V_x \cong \text{Span } \xi_x$ , so that  $\pi_*\xi_x = 0$  for all  $x \in \mathcal{M}$ , where  $\pi_* : T_x\mathcal{M} \rightarrow T_{\pi(x)}\bar{\mathcal{M}}$  designates the pushforward of the map  $\pi$  at  $x$ .

To emphasise the geometrical meaning of the ambient manifold which is the main leitmotiv of this paper, let us summarise the following facts:

**Proposition 2.2** (Ambient structure). *Let  $\mathcal{M}$  be a manifold. The following structures on  $\mathcal{M}$  are in bijective correspondence:*

1. a congruence  $I$  of parameterised open curves from  $\mathbb{R}$  to  $\mathcal{M}$ ,
2. a complete nowhere-vanishing vector field  $\xi$  on  $\mathcal{M}$ ,
3. a principal  $\mathbb{R}$ -bundle  $(\mathcal{M}, \xi)$  with fundamental vector field  $\xi$ .

*The relation between these geometrical structures is as follows: the congruence  $I$  is the family of integral curves of the vector field  $\xi$  as well as the vertical foliation of the manifold  $\mathcal{M}$ .*

Without loss of generality, such a principal  $\mathbb{R}$ -bundle  $\mathcal{M}$  can be assumed to be trivial, i.e. it is isomorphic to  $\bar{\mathcal{M}} \times \mathbb{R}$  (cf. e.g. Proposition 16.14.5 of [47]). This ensures the existence of global cross sections which can be seen as embeddings of the Platonic screen  $\bar{\mathcal{M}}$  inside the ambient manifold  $\mathcal{M}$ .

Such an ambient structure can be endowed with a notion of horizontality through the gift of a principal connection for the principal  $\mathbb{R}$ -bundle  $(\mathcal{M}, \xi)$  i.e. a 1-form  $A \in \Omega^1(\mathcal{M})$  on  $\mathcal{M}$  satisfying the following two conditions

1.  $A(\xi) = 1$
2.  $\mathcal{L}_\xi A = 0$ .

The space of principal connections will be denoted  $PC(\mathcal{M}, \xi)$ . In the following it will prove useful to relax the second condition and consider *Ehresmann connections* defined as follows:

**Definition 2.3** (Ehresmann connection). *Let  $(\mathcal{M}, \xi)$  be an ambient structure. An Ehresmann connection is a 1-form  $A \in \Omega^1(\mathcal{M})$  on  $\mathcal{M}$  satisfying  $A(\xi) = 1$ .*

Such an Ehresmann connection can be seen dual to the notion of fields of observers (*cf.* Definition 2.14 in [5]). The space of Ehresmann connections on  $(\mathcal{M}, \xi)$  will be denoted  $EC(\mathcal{M}, \xi)$ .

**Definition 2.4** (Screen worldvolume [21]). *A global section  $\sigma : \bar{\mathcal{M}} \hookrightarrow \mathcal{M}$  of a principal  $\mathbb{R}$ -bundle  $\pi : \mathcal{M} \rightarrow \bar{\mathcal{M}}$  is called a screen worldvolume.*

Let us briefly characterise the (co)tangent bundles of the Platonic screen  $\bar{\mathcal{M}}$ . More precisely, let us point out the two canonical isomorphisms of vector bundles:  $T\bar{\mathcal{M}} \cong T\mathcal{M}/\text{Span } \xi$  and  $T^*\bar{\mathcal{M}} \cong \text{Ann } \xi$ , where  $\text{Ann } \xi_x \equiv \{\alpha_x \in T_x^*\mathcal{M} \mid \alpha_x(\xi_x) = 0\}$ . The first isomorphism relies on the equivalence relation of tangent vectors to  $\mathcal{M}$  at  $x \in \mathcal{M}$ :

$$X_x \sim Y_x \quad \Leftrightarrow \quad X_x = Y_x + \lambda \xi_x, \quad \text{for some } \lambda \in \mathbb{R}. \quad (2.4)$$

Such equivalence classes are in bijective correspondence with tangent vectors to  $\bar{\mathcal{M}}$  at  $\pi(x)$ . Indeed, the kernel of the surjective pushforward  $\pi_* : T\mathcal{M} \rightarrow T\bar{\mathcal{M}}$  of the projection  $\pi : \mathcal{M} \rightarrow \bar{\mathcal{M}}$  is the vertical bundle  $\text{Span } \xi$ . The second isomorphism is provided by the pullback  $\pi^* : T^*\bar{\mathcal{M}} \rightarrow T^*\mathcal{M}$  whose image is  $\text{Ann } \xi$ .

The following three Sections intend to make use of the previous characterisation of (co)tangent bundles of  $\bar{\mathcal{M}}$  in order to discriminate among the fields living on the ambient manifold  $\mathcal{M}$  those admitting a well-defined projection on the Platonic screen  $\bar{\mathcal{M}}$ . These are standard results holding for principal bundles that we review for completeness.

## Projection and lift of a function

Given an ambient structure  $(\mathcal{M}, \xi)$ , we will call *invariant* a section  $f \in \Gamma(\otimes T\mathcal{M} \otimes \otimes T^*\mathcal{M})$  satisfying  $\mathcal{L}_\xi f = 0$  and will denote  $\Gamma_{\text{inv}}(\otimes T\mathcal{M} \otimes \otimes T^*\mathcal{M})$  the space of invariant sections of  $\otimes T\mathcal{M} \otimes \otimes T^*\mathcal{M}$ . In particular, an invariant function on the principal  $\mathbb{R}$ -bundle  $\mathcal{M}$  is a function  $f \in C^\infty(\mathcal{M})$  such that  $\mathcal{L}_\xi f = 0$ .

**Definition 2.5** (Lift of a function). *Let  $\bar{f} \in C^\infty(\bar{\mathcal{M}})$  be a function on the Platonic screen  $\bar{\mathcal{M}}$ . Via pullback by the projection map, the function  $\bar{f}$  defines a unique function  $f \equiv \pi^* \bar{f} = \bar{f} \circ \pi \in C^\infty(\mathcal{M})$  on the ambient manifold  $\mathcal{M}$ , called the lift of  $\bar{f}$ .*

There is a bijective correspondence between invariant functions on the principal  $\mathbb{R}$ -bundle  $\mathcal{M}$  and functions on the Platonic screen  $\bar{\mathcal{M}}$ , that is:  $C_{\text{inv}}^\infty(\mathcal{M}) \cong C^\infty(\bar{\mathcal{M}})$ .

## Projection and lifts of a vector field

**Definition 2.6** (Projectable and invariant vector fields). *The vector field  $X \in \Gamma(T\mathcal{M})$  on  $\mathcal{M}$  is said projectable (respectively invariant) if it satisfies  $\mathcal{L}_\xi X = f\xi$ , for some function  $f \in C^\infty(\mathcal{M})$  (respectively  $\mathcal{L}_\xi X = 0$ ).*

As suggested by its name, the pushforward  $\pi_* X \in \Gamma(T\bar{\mathcal{M}})$  by the projection map of a projectable vector field  $X \in \Gamma(T\mathcal{M})$  is a well defined vector field on the Platonic screen  $\bar{\mathcal{M}}$ .

**Proposition 2.7.** *If  $X$  and  $Y \in \Gamma(T\mathcal{M})$  are two projectable (resp. invariant) vector fields on  $\mathcal{M}$ , then their Lie bracket  $[X, Y] \in \Gamma(T\mathcal{M})$  is projectable (resp. invariant).*

**Definition 2.8** (Lift of a vector field). *Let  $\bar{X} \in \Gamma(T\bar{\mathcal{M}})$  be a vector field on the Platonic screen  $\bar{\mathcal{M}}$ . A vector field  $X \in \Gamma(T\mathcal{M})$  satisfying the conditions:*

- $X$  is projectable (resp. invariant)
- $\pi_* X = \bar{X}$

*is called a lift (resp. invariant lift) of  $\bar{X}$  in  $\mathcal{M}$ .*

Clearly, the arbitrariness in the possible lifts of a given vector field is a vertical vector field:

**Proposition 2.9.** *Let  $\bar{X} \in \Gamma(T\bar{\mathcal{M}})$  be a vector field on the Platonic screen  $\bar{\mathcal{M}}$ . Let  $X$  and  $X' \in \Gamma(T\mathcal{M})$  be two (invariant) lifts of  $\bar{X}$  in  $\mathcal{M}$ . Then, there exists a (invariant) function  $f \in C^\infty(\mathcal{M})$  such that  $X' = X + f\xi$ .*

This construction of invariant lifts provides the following isomorphism:

$$\Gamma_{\text{inv}}(T\mathcal{M}/\text{Span } \xi) \cong \Gamma(T\bar{\mathcal{M}}).$$

### Projection of a 1-form

A necessary condition in order for a 1-form  $\alpha \in \Omega^1(\mathcal{M})$  to project onto a well-defined 1-form  $\bar{\alpha} \in \Omega^1(\bar{\mathcal{M}})$  is to be a section of the vector subbundle  $\text{Ann } \xi \subset T^*\mathcal{M}$ , i.e. to satisfy  $\alpha(\xi) = 0$ .

**Proposition 2.10** (Projectable 1-form). *A 1-form  $\alpha \in \Omega^1(\mathcal{M})$  is projectable on the Platonic screen  $\bar{\mathcal{M}}$  if and only if the two following conditions are satisfied:*

1. *the 1-form annihilates the fundamental vector field, i.e.  $\alpha(\xi) = 0$ ,*
2. *the 1-form is invariant,  $\mathcal{L}_\xi \alpha = 0$ .*

*The projection  $\bar{\alpha} \in \Omega^1(\bar{\mathcal{M}})$  of the 1-form is then defined as the 1-form satisfying the relation  $\pi^*\bar{\alpha} = \alpha$ .*

This discussion explains the following isomorphism:  $\Gamma_{\text{inv}}(\text{Ann } \xi) \cong \Gamma(T^*\bar{\mathcal{M}})$ .

The previous conditions generalise straightforwardly to a covariant metric, so that the following Proposition holds:

**Proposition 2.11.** *A covariant metric  $g \in \Gamma(\vee^2 T^*\mathcal{M})$  defined on the ambient structure  $(\mathcal{M}, \xi)$  is projectable on the Platonic screen  $\bar{\mathcal{M}}$  if and only if the two following conditions are satisfied:*

1.  *$\xi \in \text{Rad } g$ , i.e.  $g(\xi) = 0$ ,*
2.  *$\mathcal{L}_\xi g = 0$ .*



If such conditions are met, the projection  $\bar{g} \in \Gamma(\vee^2 T^* \mathcal{M})$  is defined as  $\pi^* \bar{g} = g$ .

It should be noted that Proposition 2.11 eliminates any possibility to define a covariant metric on the Platonic screen of an ambient structure by projecting a Lorentzian metric, since only *degenerate* covariant metrics are projectable. In order to circumvent this drawback, we will be led to endow ambient structures with degenerate metric structures.

## 2.2 Ambient metric structures

The present Section deals with ambient structures endowed with metrical properties, not necessarily pseudo-Riemannian. We will be prominently interested with the interplay between ambient metrical structures and nonrelativistic metric structures on the Platonic screen.

From now on, we will focus on ambient structures  $(\mathcal{M}, \xi)$  endowed with an absolute clock (*cf.* Definition 2.10 in [5]), denoted  $\psi$ , which is assumed to annihilate the fundamental vector field  $\xi$ , *i.e.*  $\psi \in \Gamma(\text{Ann } \xi)$ . We start by defining an ambient analogue of the notion of absolute rulers (*cf.* Definition 2.11 in [5]).

**Definition 2.12** (Ambient absolute rulers). *A collection of ambient absolute rulers on  $\mathcal{M}$  is a field of positive semi-definite contravariant symmetric bilinear forms  $h \in \Gamma(\vee^2(\text{Ann } \xi)^*)$  acting on 1-forms annihilating  $\xi$  and such that its radical is spanned by the absolute clock  $\psi$  *i.e.**

$$\text{Rad } h \cong \text{Span } \psi. \quad (2.5)$$

*Alternatively, a collection of ambient absolute rulers can be defined as a field  $\gamma \in \Gamma(\vee^2(\text{Ker } \psi)^*)$  on  $\mathcal{M}$  of positive semi-definite covariant symmetric bilinear forms acting on tangent vectors annihilated by the absolute clock and such that the radical of  $\gamma$  is spanned by the fundamental vector field  $\xi$  *i.e.**

$$\text{Rad } \gamma \cong \text{Span } \xi. \quad (2.6)$$

**Proposition 2.13.** *The two above definitions are equivalent.*

The proof is postponed to Appendix B.

This notion of ambient absolute rulers allows to extend the nonrelativistic definition of a Leibnizian structure (cf. Definition 2.12 in [5]) to the following ambient analogue:

**Definition 2.14** (Ambient Leibnizian structure). *An ambient Leibnizian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  is a quadruplet composed by the following elements:*

- an ambient structure  $(\mathcal{M}, \xi)$ ,
- an absolute clock  $\psi \in \Gamma(\text{Ann } \xi)$  on  $\mathcal{M}$ ,
- a collection of ambient absolute rulers  $\gamma \in \Gamma(\vee^2(\text{Ker } \psi)^*)$ .

Pursuing the analogy with the nonrelativistic case, we define the two following subclasses (cf. Table 1 in [5]):

- An *ambient Aristotelian* structure will designate an ambient Leibnizian structure whose absolute clock satisfies the Frobenius integrability condition (i.e.  $\psi \wedge d\psi = 0$ ).
- An ambient Leibnizian structure with closed absolute clock (i.e.  $d\psi = 0$ ) will be called an *ambient Augustinian* structure.

The Frobenius integrability condition enjoyed by the absolute clock of any ambient Aristotelian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  ensures that the kernel of  $\psi$  induces an *involutive* distribution  $\mathcal{D} = \{\mathcal{D}_x | \forall x \in \mathcal{M}\}$ , with  $\mathcal{D}_x \equiv \text{Ker } \psi_x$ . The ambient spacetime  $\mathcal{M}$  is thus foliated by codimension-one hypersurfaces. Each leaf  $\tilde{\mathcal{M}}$  of the foliation is a maximal integral submanifold of  $\mathcal{M}$  for the distribution  $\mathcal{D}$  and is characterised by an immersion  $i : \tilde{\mathcal{M}} \hookrightarrow \mathcal{M}$  satisfying the properties:

- $i_* (T\tilde{\mathcal{M}}) \cong \text{Ker } \psi$
- $\text{Ker } i^* \cong \text{Span } \psi$ .

Any ambient Aristotelian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  then induces a Carrollian structure  $\mathcal{C}(\tilde{\mathcal{M}}, \tilde{\xi}, \tilde{\gamma})$  (cf. Definition A.2) on each leaf of the foliation, where  $i_* \tilde{\xi} \equiv \xi$  and  $\tilde{\gamma} \equiv i^* \gamma$ . Note that the induced Carrollian structure is the most general one can defined. In particular, the present construction allows to embed non-invariant Carrollian structures (i.e. with  $\mathcal{L}_{\tilde{\xi}} \tilde{\gamma} \neq 0$ ) inside an ambient spacetime.

An ambient Leibnizian structure will be said projectable if both the absolute clock and rulers are projectable. Since both  $\psi$  and  $\gamma$  have been assumed to annihilate the fundamental vector field  $\xi$ , the only remaining condition consists in imposing their invariance, *i.e.*  $\mathcal{L}_\xi \psi = \mathcal{L}_\xi \gamma = 0$ . The following Proposition justifies further the terminology used:

**Proposition 2.15.** *Let  $(\mathcal{M}, \xi)$  be an ambient structure. Projectable ambient Leibnizian structures  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  are in bijective correspondence with Leibnizian structures  $\bar{\mathcal{L}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  on the Platonic screen  $\bar{\mathcal{M}}$ .*

The proof is straightforward and follows from the fact that the pullback  $\pi^*$  of the projection map  $\pi : \mathcal{M} \rightarrow \bar{\mathcal{M}}$  defines the two isomorphisms  $\Gamma(T^*\bar{\mathcal{M}}) \cong \Gamma_{\text{inv}}(\text{Ann } \xi)$  and  $\Gamma(\vee^2(\text{Ker } \bar{\psi})^*) \cong \Gamma_{\text{inv}}(\vee^2(\text{Ker } \psi)^*) \cap \Gamma_{\text{inv}}(\vee^2 \text{Ann } \xi)$ .

**Example 2.16** (Ambient Aristotle spacetime). The most simple example of an ambient Augustinian structure is given by an ambient spacetime  $\mathcal{M} \cong \mathbb{R}^{d+2}$  with coordinates  $(u, t, x^i)$  characterised by a fundamental vector field  $\xi = \partial/\partial u$ , a closed absolute clock  $\psi = dt$  and flat ambient absolute rulers  $\gamma = \delta_{ij} dx^i \vee dx^j$ . On the one hand, this structure projects on the Platonic screen as the  $(d+1)$ -dimensional Aristotle spacetime  $\bar{\mathcal{M}} \cong \mathbb{R}^{d+1}$  with coordinates  $(t, x^i)$  (*cf.* Example 2.20 of [5]). On the other hand, the hyperplanes  $t = \text{cst}$  of the ambient Aristotle spacetime are Carroll spacetimes (*cf.* Example A.3) with coordinates  $(u, x^i)$ .

**Proposition 2.17** (Leibniz algebra). *The infinite-dimensional algebra  $\mathfrak{leib}_\infty(d+2)$  of infinitesimal isometries of the ambient Aristotle spacetime  $(\mathcal{M}, \xi, \psi, \gamma)$ , *i.e.* the algebra of vector fields  $X \in \Gamma(T\mathcal{M})$  satisfying*

$$\mathcal{L}_X \xi = \mathcal{L}_X \psi = \mathcal{L}_X \gamma = 0 \quad (2.7)$$

*is spanned by vector fields of the form*

$$X = a(t, x^i) \frac{\partial}{\partial u} + \tau \frac{\partial}{\partial t} + [\lambda^i_j(t) x^j + b^i(t)] \frac{\partial}{\partial x^i} \quad (2.8)$$

*where  $a$  is an arbitrary function of  $t$  and  $x^i$  while  $\lambda$  (resp.  $b$ ) are arbitrary  $\mathfrak{o}(d)$  (resp.  $\mathbb{R}^d$ ) valued functions of  $t$  only.*

*The finite-dimensional subalgebra  $\mathfrak{leib}(d+2) = \mathfrak{leib}_\infty(d+2) \cap \mathfrak{igl}(d+2)$  of affine infinitesimal isometries of the ambient Aristotle spacetime  $(\mathcal{M}, \xi, \psi, \gamma)$ , *i.e.* the alge-*

bra of vector fields  $X \in \Gamma(T\mathcal{M})$  satisfying (2.7) and whose second partial derivatives vanish, is generated by the following vector fields:

- *Mass:*  $M \equiv \partial_u$
- *Galilei Hamiltonian:*  $H \equiv \partial_t$
- *Translation*  $P_i \equiv \partial_i$
- *Carroll Hamiltonian:*  $C \equiv t \partial_u$
- *Carroll boosts:*  $D_i \equiv x_i \partial_u$
- *Galilei boosts:*  $K_i \equiv x_i \partial_u - t \partial_i$
- *Rotations:*  $J_{ij} \equiv x_i \partial_j - x_j \partial_i$ .

The previous generators satisfy the (schematic) commutation relations:

$$\begin{aligned} [H, C] &\sim M, & [P, K] &\sim M, & [P, D] &\sim M \\ [H, K] &\sim P, & [D, K] &\sim C, & [P, J] &\sim P \\ [K, J] &\sim K, & [D, J] &\sim D, & [J, J] &\sim J. \end{aligned}$$

It can be checked from the previous commutation relations that the finite-dimensional subalgebra  $\mathfrak{leib}(d+2)$  can be written as the semidirect<sup>3</sup> sum

$$\mathfrak{leib}(d+2) = \mathfrak{carr}(d+1) \in \mathbb{R}^{d+2}$$

with  $\mathfrak{carr}(d+1) \cong \text{Span} \{C, D_i, K_i, J_{ij}\}$  the (inhomogeneous) Carroll algebra and  $\mathbb{R}^{d+2} \cong \text{Span} \{H, M, P_i\}$  an Abelian ideal.

In the above realisation, the Carroll Hamiltonian  $C$  and Galilei boosts  $K_i$  become trivial on the leaf  $t = 0$ . On any other leaf of constant  $t \neq 0$ , the same generators take the interpretation of the Galilei Hamiltonian  $H$  and spatial translations  $P_i$ .

Like its base counterpart, a collection of absolute rulers  $\gamma$  (resp.  $h$ ) is not defined on the whole (co)tangent space. One can circumvent this drawback and define, though

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<sup>3</sup>Let  $\mathfrak{g}$  be a Lie algebra with  $\mathfrak{h} \subseteq \mathfrak{g}$  a Lie subalgebra (i.e.  $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ ) and  $\mathfrak{i} \subseteq \mathfrak{g}$  an ideal (i.e.  $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$ ). The algebra  $\mathfrak{g}$  is the semidirect sum of  $\mathfrak{i}$  with  $\mathfrak{h}$  if it admits the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{i}$  as  $\mathfrak{h}$ -modules. The subalgebra  $\mathfrak{h}$  will be called the *homogeneous* part of  $\mathfrak{g}$ .

in a non-canonical way, a covariant (resp. contravariant) metric by making use of additional structures, namely a field of observers (resp. an Ehresmann connection).

We start by defining various projectors as follows:

Given a field of observers  $N \in FO(\mathcal{M}, \psi)$  (i.e.  $N \in \Gamma(T\mathcal{M})$  and  $\psi(N) = 1$ ), one can define a field of endomorphisms  $P^N : \Gamma(T\mathcal{M}) \rightarrow \Gamma(\text{Ker } \psi)$  as

$$P^N(X) = X - \psi(X)N, \text{ with } X \in \Gamma(T\mathcal{M}) \text{ a vector field on } \mathcal{M} \quad (2.9)$$

together with its transpose  $\bar{P}^N : \Omega^1(\mathcal{M}) \rightarrow \Gamma(\text{Ann } N)$

$$\bar{P}^N(\alpha) = \alpha - \alpha(N)\psi, \text{ with } \alpha \in \Omega^1(\mathcal{M}) \text{ a 1-form on } \mathcal{M}. \quad (2.10)$$

Their respective kernels are given by  $\text{Ker } P^N \cong \text{Span}\{N\}$  and  $\text{Ker } \bar{P}^N \cong \text{Span}\{\psi\}$ . Finally, the gift of an Ehresmann connection  $A \in EC(\mathcal{M}, \xi)$  allows to define the field of endomorphisms  $P^A : \Gamma(T\mathcal{M}) \rightarrow \Gamma(\text{Ker } A)$  as

$$P^A(X) = X - A(X)\xi, \text{ with } X \in \Gamma(T\mathcal{M}) \text{ a vector field on } \mathcal{M} \quad (2.11)$$

while its transpose reads  $\bar{P}^A : \Omega^1(\mathcal{M}) \rightarrow \Gamma(\text{Ann } \xi)$

$$\bar{P}^A(\alpha) = \alpha - \alpha(\xi)A, \text{ with } \alpha \in \Omega^1(\mathcal{M}) \text{ a 1-form on } \mathcal{M} \quad (2.12)$$

whose respective kernels read  $\text{Ker } P^A \cong \text{Span}\{\xi\}$  and  $\text{Ker } \bar{P}^A \cong \text{Span}\{A\}$ .

We will now start discussing bases adapted to the ambient metric structure. While discussing bases at a given point, we will not write explicitly the point  $x \in \mathcal{M}$ . Via a slight abuse of notation, this allows to make the extension to field of bases.

**Definition 2.18** (Leibnizian basis). *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be an ambient Leibnizian structure. A Leibnizian basis of the tangent space  $T_x\mathcal{M}$  at a point  $x \in \mathcal{M}$  is an ordered basis  $L = \{\xi, N, e_1, \dots, e_d\}$  with  $\xi$  the fundamental vector,  $N$  the tangent vector of an observer,  $\{\xi, e_1, \dots, e_d\}$  a basis of  $\text{Ker } \psi$ , and  $\{e_1, \dots, e_d\}$  an orthonormal system with respect to  $\gamma$ .*

Explicitly, the basis  $L = \{\xi, N, e_1, \dots, e_d\}$  must satisfy the conditions:

1.  $\psi(\xi) = 0, \psi(N) = 1, \psi(e_i) = 0$

$$2. \gamma(e_i, e_j) = \delta_{ij}$$

**Proposition 2.19.** *The set of automorphisms of  $T_x \mathcal{M}$  preserving the collection of Leibnizian bases at a point forms a group isomorphic to the inhomogeneous Carroll group  $\text{Carr}(d+1) \subset GL(d+2, \mathbb{R})$  defined by the following set of matrices:*

$$T = \begin{pmatrix} 1 & a & -\mathbf{f}^T \mathbf{R} \\ 0 & 1 & 0 \\ 0 & \mathbf{b} & \mathbf{R} \end{pmatrix} \quad (2.13)$$

with  $a \in \mathbb{R}$ ,  $\mathbf{b}, \mathbf{f} \in \mathbb{R}^d$  and  $\mathbf{R} \in O(d)$ .

The Carroll group [38, 39] admits a semidirect product structure decomposition as  $\text{Carr}(d+1) = O(d) \ltimes H(d)$  where  $O(d)$  stands for the  $d$ -dimensional orthogonal group ( $a = 0$ ,  $\mathbf{b} = \mathbf{0} = \mathbf{f}$ ) and  $H(d)$  for the  $d$ -dimensional Heisenberg group ( $\mathbf{R} = \mathbb{1}$ ). The Carroll group acts regularly on the space of Leibnizian bases at  $x$  via the group action:

$$\{\xi, N, e_i\} \mapsto \{\xi, N + \mathbf{b}^i e_i + a \xi, \mathbf{R}^j_i (e_j - \mathbf{f}_j^T \xi)\}. \quad (2.14)$$

so that the space of Leibnizian bases at a point is a principal homogeneous space of the Carroll group. Consequently, the bundle of Leibnizian bases, denoted by  $LB(\mathcal{M}, \xi, \psi, \gamma)$ , is a  $\text{Carr}(d+1)$ -structure on  $\mathcal{M}$ . A converse statement can be formulated by asserting that the base space of a  $\text{Carr}(d+1)$ -structure is an ambient Leibnizian structure.

A basis of  $T_x^* \mathcal{M}$  dual to  $L = \{\xi, N, e_i\}$  can be defined as  $L^* \equiv \{A, \psi, \theta^i\}$  where

$$3. A(\xi) = 1, A(N) = 0, A(e_i) = 0$$

$$4. \theta^i(\xi) = 0, \theta^i(N) = 0, \theta^i(e_j) = \delta_j^i.$$

Remember that the condition  $A(\xi) = 1$  holds by definition for an Ehresmann connection. All together, the conditions 1-4 state that  $L^*$  is indeed the dual basis to  $L$ . Therefore it is clear that the Carroll group acts regularly on the space of dual

Leibnizian bases at a point. More explicitly, the group action reads:

$$\{A, \psi, \theta^i\} \mapsto \{A + \mathbf{f}_i^T \theta^i - (a + \mathbf{f}_i^T \mathbf{b}^i) \psi, \psi, \mathbf{R}^{Ti}{}_j (\theta^j - \mathbf{b}^j \psi)\}. \quad (2.15)$$

As a last piece of terminology, we introduce the notion of Leibnizian pair:

**Definition 2.20** (Leibnizian pair). *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be an ambient Leibnizian structure and  $LB(\mathcal{M}, \xi, \psi, \gamma)/O(d)$  be the space of  $O(d)$ -orbits acting on the bundle of Leibnizian bases. A section of the former bundle will be called a Leibnizian pair. The space of Leibnizian pairs will be denoted*

$$LP(\mathcal{M}, \xi, \psi) \equiv \Gamma \left( \frac{LB(\mathcal{M}, \xi, \psi, \gamma)}{O(d)} \right).$$

Explicitly, a Leibnizian pair is a couple  $(N, A)$  with  $N \in FO(\mathcal{M}, \psi)$  a field of observers and  $A \in EC(\mathcal{M}, \xi)$  an Ehresmann connection on  $\mathcal{M}$  satisfying  $A(N) = 0$ . The typical fiber of the bundle  $LB(\mathcal{M}, \xi, \psi, \gamma)/O(d)$  over  $\mathcal{M}$  is the coset space  $\text{Carr}(d+1)/O(d) \cong H(d)$  which is of dimension  $2d+1$  and therefore matches the number of independent components of a Leibnizian pair:  $2(d+1) - 1$ .

We will denote by  $\mathcal{H}(d)$  the group  $\mathcal{H}(d) \equiv (\Gamma(\text{Ker } \psi / \text{Span } \xi) \oplus \Gamma(\text{Ann } \xi / \text{Span } \psi)) \ltimes C^\infty(\mathcal{M})$  endowed with the composition law

$$([V'], [\alpha'], a') \cdot ([V], [\alpha], a) = ([V' + V], [\alpha' + \alpha], a' + a - \alpha(V'))$$

where  $[V] \in \Gamma(\text{Ker } \psi / \text{Span } \xi)$  denotes an equivalence class of vector fields  $V \in \Gamma(\text{Ker } \psi)$  differing by a multiple of the fundamental vector field  $\xi$  and  $[\alpha] \in \Gamma(\text{Ann } \xi / \text{Span } \psi)$  stands for an equivalence class of 1-forms  $\alpha \in \Gamma(\text{Ann } \xi)$  differing by a multiple of the absolute clock  $\psi$ .

The group  $\mathcal{H}(d)$  corresponds to the local action of the Heisenberg group  $H(d)$  on the space of Leibnizian pairs  $LP(\mathcal{M}, \xi, \psi)$  via

$$\begin{aligned} \rho : \quad LP(\mathcal{M}, \xi, \psi) \times \mathcal{H}(d) &\rightarrow LP(\mathcal{M}, \xi, \psi) \\ ((N, A), ([V], [\alpha], a)) &\mapsto (N + P^A(V) + a\xi, A + \bar{P}^N(\alpha) - (a + \alpha(V))\psi) \end{aligned} \quad (2.16)$$

where  $V \in \Gamma(\text{Ker } \psi)$  and  $\alpha \in \Gamma(\text{Ann } \xi)$  are representatives of the equivalence classes  $[V] \in \Gamma(\text{Ker } \psi / \text{Span } \xi)$  and  $[\alpha] \in \Gamma(\text{Ann } \xi / \text{Span } \psi)$ , respectively. Since

the previous action of the local Heisenberg group  $\mathcal{H}(d)$  is regular, the space of Leibnizian pairs  $LP(\mathcal{M}, \xi, \psi)$  is a principal homogeneous space. The notion of Leibnizian pair allows to define:

**Definition 2.21** (Spacelike projection of vector fields). *Let  $L \equiv (N, A) \in LP(\mathcal{M}, \xi, \psi)$  be a Leibnizian pair. The field of endomorphisms  $P^L : \Gamma(T\mathcal{M}) \rightarrow \Gamma(\text{Ker } \psi \cap \text{Ker } A)$  defined as*

$$P^L(X) = X - \psi(X)N - A(X)\xi \quad (2.17)$$

with  $X \in \Gamma(T\mathcal{M})$  a vector field on  $\mathcal{M}$ , is called a spacelike projector of vector fields.

Note that  $\text{Ker } P^L \cong \text{Span}\{\xi, N\}$ . The transpose of  $P^L$ , denoted  $\bar{P}^L : \Omega^1(\mathcal{M}) \rightarrow \Gamma(\text{Ann } \xi \cap \text{Ann } N)$  is given by

$$\bar{P}^L(\alpha) = \alpha - \alpha(N)\psi - \alpha(\xi)A \quad (2.18)$$

and  $\text{Ker } \bar{P}^L \cong \text{Span}\{\psi, A\}$ .

**Definition 2.22** (Ambient transverse metrics). *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be an ambient Leibnizian structure and  $N \in FO(\mathcal{M}, \psi)$  a field of observers. The covariant ambient transverse metric  $\overset{N}{\gamma} \in \Gamma(\vee^2 T^*\mathcal{M})$  is defined by its action on vector fields  $X, Y \in \Gamma(TM)$  as*

$$\overset{N}{\gamma}(X, Y) = \gamma(P^N(X), P^N(Y)) \quad (2.19)$$

where  $P^N : \Gamma(T\mathcal{M}) \rightarrow \Gamma(\text{Ker } \psi)$  stands for the projector on  $\text{Ker } \psi$  associated to the field of observers  $N$ .

Conversely, the contravariant transverse metric  $\overset{A}{h} \in \Gamma(\vee^2 T\mathcal{M})$  can be defined by its action on 1-forms  $\alpha, \beta \in \Omega^1(\mathcal{M})$  as

$$\overset{A}{h}(\alpha, \beta) = h(\bar{P}^A(\alpha), \bar{P}^A(\beta)). \quad (2.20)$$

where  $\bar{P}^A : \Omega^1(\mathcal{M}) \rightarrow \Gamma(\text{Ann } \xi)$  is the projector on  $\text{Ann } \xi$  associated to the Ehresmann connection  $A$ .



The following relation follows from the previous definitions:

$$h^{\mu\lambda} \gamma_{\lambda\nu}^N = P_{\nu}^{L\mu} = \delta_{\nu}^{\mu} - \xi^{\mu} A_{\nu} - N^{\mu} \psi_{\nu}.$$

We now investigate how ambient Leibnizian structures can be induced by Lorentzian spacetimes. We first introduce the notion of fibered Lorentzian structures :

**Definition 2.23** (Fibered Lorentzian structure). *A fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  is a triplet composed by*

- *an ambient structure  $(\mathcal{M}, \xi)$*
- *a Lorentzian metric  $g \in \Gamma(\vee^2 T^* \mathcal{M})$  on  $\mathcal{M}$  for which the fundamental vector field  $\xi$  is null.*

We will refer to the 1-form  $\psi \in \Omega^1(\mathcal{M})$  dual to  $\xi$  via  $g$  (*i.e.*  $\psi \equiv g(\xi)$ ) as the *fundamental covector field*. Note that, being a nowhere vanishing 1-form (since its dual  $\xi$  is nowhere-vanishing and  $g$  is nondegenerate), the fundamental covector field  $\psi$  defines an absolute clock on  $\mathcal{M}$ . Furthermore, since the fundamental vector field  $\xi$  is light-like, the absolute clock annihilates  $\xi$ , *i.e.*  $\psi \in \Gamma(\text{Ann } \xi)$ , or equivalently,  $\xi \in \Gamma(\text{Ker } \psi)$ .

A collection of ambient rulers  $\gamma$  can now be constructed by restricting the Lorentzian metric  $g \in \Gamma(\vee^2 T^* \mathcal{M})$  to the subbundle  $\text{Ker } \psi$ . Explicitly, we define the field  $\gamma \in \Gamma(\vee^2 (\text{Ker } \psi)^*)$  via its action as:

$$\gamma(V, W) = g(V, W) \tag{2.21}$$

for all vector fields  $V, W \in \Gamma(\text{Ker } \psi)$ . Note that  $\xi \in \text{Rad } \gamma$ , since  $\gamma(V, \xi) = g(V, \xi) = \psi(V) = 0$  for all  $V \in \Gamma(\text{Ker } \psi)$ . The fact that  $g$  is Lorentzian then guarantees that  $\text{Rad } \gamma \cong \text{Span } \xi$ , so that  $\gamma$  fits the definition of a collection of absolute rulers. The following Proposition sums up the previous discussion:

**Proposition 2.24.** *Any fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  induces an ambient Leibnizian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  as:*

1.  $\psi \equiv g(\xi)$
2.  $\gamma \equiv g|_{\text{Ker } \psi}$ .

The fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  is then said to complement the ambient Leibnizian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$ .

A converse statement will be provided in Proposition 2.31 where we will identify the necessary structure supplementing an ambient Leibnizian structure in order to uniquely determine a fibered Lorentzian structure that complements it. In the meantime, we investigate the subclass of fibered Lorentzian structures inducing projectable ambient Leibnizian structures.

**Proposition 2.25.** *A fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  induces a projectable ambient Leibnizian structure if and only if the two following conditions are met:*

- $\xi \in \text{Rad } \mathcal{L}_\xi g$
- $(\mathcal{L}_\xi g)(V, W) = 0$ , for all  $V, W \in \Gamma(\text{Ker } \psi)$ .

An obvious corollary is that  $\xi$  being a Killing vector field of the metric  $g$  is a sufficient (but not necessary) condition for a fibered Lorentzian structure to define a projectable ambient Leibnizian structure.

**Proof:**

It has been argued earlier that both the absolute clock and rulers annihilate the fundamental vector field  $\xi$ , so what remains to be proved is the invariance of the ambient Leibnizian structure, *i.e.*  $\mathcal{L}_\xi \psi = 0 = \mathcal{L}_\xi \gamma$ . The condition that  $\xi$  belongs to  $\text{Rad } \mathcal{L}_\xi g$  can be rewritten as  $(\mathcal{L}_\xi g)(\xi) = \mathcal{L}_\xi(g(\xi)) = \mathcal{L}_\xi \psi = 0$ . Furthermore, using Cartan's formula, one can deduce from the invariance of  $\psi$  that, for all  $V \in \Gamma(\text{Ker } \psi)$ ,  $(\mathcal{L}_\xi \psi)(V) = d\psi(\xi, V) = \psi([V, \xi]) = 0$ , so that

$$[\xi, V] \in \Gamma(\text{Ker } \psi), \text{ for all } V \in \Gamma(\text{Ker } \psi). \quad (2.22)$$

This last condition guarantees that the following equivalence holds whenever  $\mathcal{L}_\xi \psi = 0$ :

$$(\mathcal{L}_\xi g)(V, W) = 0 \text{ for all } V, W \in \Gamma(\text{Ker } \psi) \iff \mathcal{L}_\xi \gamma = 0 \quad (2.23)$$

where  $\gamma \in \Gamma(\vee^2(\text{Ker } \psi)^*)$  is defined as the restriction of the Lorentzian metric  $g$  to the subbundle  $\text{Ker } \psi$  (*cf.* eq.(2.21)). This is easily shown from the

following argument:

$$\begin{aligned}
(\mathcal{L}_\xi g)(V, W) &= \xi[g(V, W)] - g([\xi, V], W) - g(V, [\xi, W]) \\
&= \xi[\gamma(V, W)] - \gamma([\xi, V], W) - \gamma(V, [\xi, W]) \\
&= (\mathcal{L}_\xi \gamma)(V, W).
\end{aligned}$$

Note that condition (2.22) is necessary in order for the Lie derivative  $\mathcal{L}_\xi \gamma$  to be well-defined.  $\square$

The previous Proposition thus singles out the necessary and sufficient conditions in order for a a fibered Lorentzian structure to induce a Leibnizian structure on its Platonic screen.

**Example 2.26** (Bargmann spacetime). The simplest example of a fibered Lorentzian structure is given by an ambient spacetime  $\mathcal{M} \cong \mathbb{R}^{d+2}$  with coordinates  $(u, t, x^i)$  characterised by the fundamental vector field  $\xi = \partial/\partial u$  and the Minkowski metric  $\eta$  (with only non-vanishing components  $\eta_{ut} = 1$ ,  $\eta_{ij} = \delta_{ij}$ ). The Bargmann spacetime induces the ambient Aristotle spacetime  $(\mathcal{M}, \xi, \psi, \gamma)$  of Example 2.16 with  $\psi \equiv \eta(\xi)$  and  $\gamma \equiv \eta|_{\text{Ker } \psi}$ . The algebra of infinitesimal isometries of the Bargmann spacetime is the (finite-dimensional) Bargmann algebra  $\mathfrak{bar}(d+2)$ . The Bargmann algebra is the subalgebra of the Leibniz algebra  $\mathfrak{leib}(d+2)$  generated by  $\text{Span } \{H, M, P_i, K_i, J_{ij}\}$ , since Carroll Hamiltonian  $C$  and boosts  $D_i$  do not preserve the Minkowski metric.

Before characterising more explicitly the class of fibered Lorentzian structures inducing nonrelativistic metric structures on their Platonic screen, we introduce a subclass of Leibnizian bases as follows:

**Definition 2.27** (Bargmann basis). *Let  $(\mathcal{M}, \xi, g)$  be a fibered Lorentzian structure inducing the ambient Leibnizian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$ . A Bargmann basis  $B$  at a point  $x \in \mathcal{M}$  is a Leibnizian basis  $\{\xi, N, e_i\}$  and  $\{A, \psi, \theta^i\}$  such that  $A = g(N)$ .*

In other words, a Bargmann basis is a “light-cone” basis with respect to the Lorentzian metric  $g$ . This additional condition reduces the group action on the space of Bargmann bases at  $x$  from the Carroll group to the homogeneous Galilei group  $\text{Gal}_0(d+1) \subset \text{Carr}(d+1)$  in the  $d+2$ -dimensional faithful representation inher-

ited from that of the Bargmann group<sup>4</sup> defined by the following set of matrices

$$T = \begin{pmatrix} 1 & -\frac{1}{2}\mathbf{b}^T\mathbf{b} & -\mathbf{b}^T\mathbf{R} \\ 0 & 1 & 0 \\ 0 & \mathbf{b} & \mathbf{R} \end{pmatrix} \quad (2.24)$$

with  $\mathbf{b} \in \mathbb{R}^d$  and  $\mathbf{R} \in O(d)$ . The homogeneous Galilei group therefore acts regularly on the space of Bargmann bases via the group action:

$$\{\xi, N, e_i\} \mapsto \left\{ \xi, N + \mathbf{b}^i e_i - \frac{1}{2}\mathbf{b}_i^T \mathbf{b}^i \xi, \mathbf{R}_i^j (e_j - \mathbf{b}_j^T \xi) \right\} \quad (2.25)$$

$$\{A, \psi, \theta^i\} \mapsto \left\{ A + \mathbf{b}_i^T \theta^i - \frac{1}{2}\mathbf{b}_i^T \mathbf{b}^i \psi, \psi, \mathbf{R}^{Ti}{}_j (\theta^j - \mathbf{b}^j \psi) \right\}. \quad (2.26)$$

In other words, the group permuting the light-cone bases of a Lorentzian spacetime while preserving one of the light-like direction is the homogeneous Galilei group, in its Bargmann representation. The origin of this fact can be traced back to the embedding of the Bargmann group inside the Poincaré group (*cf.* [48]).

The analogue of a Leibnizian pair in the Bargmann case (*i.e.* an orbit of the orthogonal group  $O(d) \subset \text{Gal}_0(d+1)$  acting on the space of Bargmann bases) identifies with the notion of a field of light-like observers (*i.e.*  $N \in FO(\mathcal{M}, \psi)$  and  $g(N, N) = 0$ ). The Abelian subgroup  $\mathbb{R}^d \subset \text{Gal}_0(d+1)$  defined by (2.24) with  $\mathbf{R} = \mathbf{1}$  is a normal subgroup of the homogeneous Carroll group. The local action of this normal subgroup on fields of Bargmann bases  $B = \{\xi, N, e_i\}$  defines the regular action of  $\Gamma(\text{Ker } \psi / \text{Span } \xi)$ , called *ambient Milne group*, on the space  $FLO(\mathcal{M}, \psi, g)$  of fields of light-like observers via:

$$N \mapsto N + P^A(V) - \frac{1}{2}\gamma(V, V)\xi$$

with  $V \in \Gamma(\text{Ker } \psi)$  a representative of  $\Gamma(\text{Ker } \psi / \text{Span } \xi)$  and  $A \in EC(\mathcal{M}, \xi)$  the Ehresmann connection dual to  $N$  *i.e.*  $A \equiv g(N)$ . Since the ambient Milne group  $\Gamma(\text{Ker } \psi / \text{Span } \xi)$  acts regularly on  $FLO(\mathcal{M}, \psi, g)$ , the space of field of light-like observers is an affine space modelled on  $\Gamma(\text{Ker } \psi / \text{Span } \xi)$ . The previous group ac-

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<sup>4</sup>We refer to Section II.A of [22] for more details.

tion also induces an action of  $\Gamma(\text{Ker } \psi / \text{Span } \xi)$  on the space of light-like Ehresmann connections ( $A \in EC(\mathcal{M}, \xi)$  and  $g^{-1}(A, A) = 0$ ) as

$$A \mapsto A + \frac{N}{\gamma}(V) - \frac{1}{2}\gamma(V, V)\psi$$

where  $N \in FLO(\mathcal{M}, \psi, g)$  is the field of light-like observers dual to  $A$ . Again, since this action is regular, the space of light-like Ehresmann connections is an affine space modelled on the ambient Milne group  $\Gamma(\text{Ker } \psi / \text{Span } \xi)$ .

These two group actions can now be used in order to define a group action  $\tau : (LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ker } \psi / \text{Span } \xi)) \rightarrow LP(\mathcal{M}, \xi, \psi)$  of the ambient Milne group  $\Gamma(\text{Ker } \psi / \text{Span } \xi)$  on the space  $LP(\mathcal{M}, \xi, \psi)$  of Leibnizian pairs as

$$(N, A) \mapsto \left( N + P^A(V) - \frac{1}{2}\gamma(V, V)\xi, A + \frac{N}{\gamma}(V) - \frac{1}{2}\gamma(V, V)\psi \right). \quad (2.27)$$

**Definition 2.28** (Ambient gravitational potential). *An ambient gravitational potential is a  $\Gamma(\text{Ker } \psi / \text{Span } \xi)$ -orbit on the space  $LP(\mathcal{M}, \xi, \psi)$  of Leibnizian pairs under the group action (2.27). The space of ambient gravitational potentials will be denoted*

$$\mathcal{P}(\mathcal{M}, \xi, \psi, \gamma) \equiv \frac{LP(\mathcal{M}, \xi, \psi)}{\Gamma(\text{Ker } \psi / \text{Span } \xi)}.$$

**Proposition 2.29.** *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be a projectable ambient Leibnizian structure and denote  $\bar{\mathcal{L}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  its projection on the Platonic screen  $\bar{\mathcal{M}}$ . The affine space of invariant ambient gravitational potentials  $\mathcal{P}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  on  $\mathcal{M}$  is isomorphic to the affine space of gravitational potentials  $\bar{\mathcal{P}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$ .*

A definition of nonrelativistic gravitational potentials is given by Definition 3.19 in [5]. The bijective map reads

$$\mathcal{P}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \bar{\mathcal{P}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma}) : [N, A] \mapsto [\bar{N}, [\bar{A}]] \quad (2.28)$$

where  $\bar{N} \equiv \pi_* N$  and  $[\bar{A}]$  is the set of gauge connections  $\bar{A} \equiv \sigma^* A$ , with  $\sigma : \bar{\mathcal{M}} \rightarrow \mathcal{M}$  an arbitrary section. The map  $A \mapsto [\bar{A}]$  is therefore bijective and given  $\bar{N}$ , the ambient field of observers  $N$  can be defined as the unique horizontal lift of  $\bar{N}$  with respect to  $A$ . The fact that the action (2.27) projects onto the action (3.36) of [5] concludes the proof.

We note that the definition of ambient gravitational potentials does not involve a notion of Lorentzian metric. In fact, ambient gravitational potentials constitute the necessary structure supplementing an ambient Leibnizian structure in order to define a Lorentzian metric complementing the latter. In order to give a precise meaning to this assertion, we introduce the notion of ambient Lagrangian metric as follows:

**Definition 2.30** (Ambient Lagrangian metric). *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be an ambient Leibnizian structure. An ambient Lagrangian metric is a covariant metric  $g \in \Gamma(\vee^2 T^* \mathcal{M})$  of Lorentzian signature satisfying*

$$\begin{cases} g(\xi) = \psi \\ \gamma \equiv g|_{\text{Ker } \psi}. \end{cases} \quad (2.29)$$

*The space of ambient Lagrangian metrics will be denoted  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$ .*

We are now in a position to formulate the following Proposition:

**Proposition 2.31.** *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be an ambient Leibnizian structure. The space  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  of ambient Lagrangian metrics forms an affine space canonically isomorphic to the affine space  $\mathcal{P}(\mathcal{M}, \xi, \psi, \gamma)$  of ambient gravitational potentials.*

In other words, given an ambient Leibnizian structure, any complementing Lorentzian metric is uniquely determined by the gift of an ambient gravitational potential. A proof of the previous Proposition can be found in Appendix B. We conclude the present section by displaying the analogue of Proposition 2.29 for Lagrangian metrics:

**Proposition 2.32.** *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be a projectable ambient Leibnizian structure and denote  $\bar{\mathcal{L}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  its projection on the Platonic screen  $\bar{\mathcal{M}}$ . The affine space of invariant ambient Lagrangian metrics  $\mathcal{L}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  on  $\mathcal{M}$  is isomorphic to the affine space of Lagrangian structures  $\bar{\mathcal{L}}(\bar{\mathcal{M}}, \bar{\psi}, [\bar{g}])$  compatible with  $\bar{\mathcal{L}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$ .*

Nonrelativistic Lagrangian structures are defined in Definition 3.30 in [5]. The proof is straightforward using Proposition 2.31 and its nonrelativistic avatar (Proposition 3.29 in [5]) as well as Proposition 2.29. Explicitly, the bijective map is given by  $g \mapsto [\bar{g}]$  where  $\bar{g} \equiv \sigma^* g$  with  $\sigma : \bar{\mathcal{M}} \rightarrow \mathcal{M}$  an arbitrary section.

## 2.3 Gravitational waves

Before addressing the issue of ambient Galilean connections, we conclude the present Section by investigating a special class of fibered Lorentzian structures:

**Definition 2.33** (Gravitational wave [21]). *A gravitational wave  $(\mathcal{M}, \xi, g)$  is a fibered Lorentzian structure whose fundamental vector field  $\xi \in \Gamma(T\mathcal{M})$  is hypersurface-orthogonal. The vector field  $\xi$  is then called the wave vector field while its dual  $\psi \equiv g(\xi)$  is referred to as the wave covector field.*

More precisely, since  $\xi$  is hypersurface-orthogonal, the wave covector field  $\psi$  is twistless (*i.e.*  $\psi$  satisfies the Frobenius integrability condition  $\psi \wedge d\psi = 0$ ) and so defines an absolute time (*cf.* Section 2.4 in [5]) on  $\mathcal{M}$ . This absolute time is usually called the “retarded time” in the gravitational wave interpretation. Moreover, in this interpretation the integral curves and hypersurfaces are designated as follows:

**Definition 2.34** (Rays and wavefront worldvolumes [21]). *The integral curves of the wave vector field are called rays. The hypersurfaces orthogonal to the congruence of rays of a gravitational wave are designated as wavefront worldvolumes.*

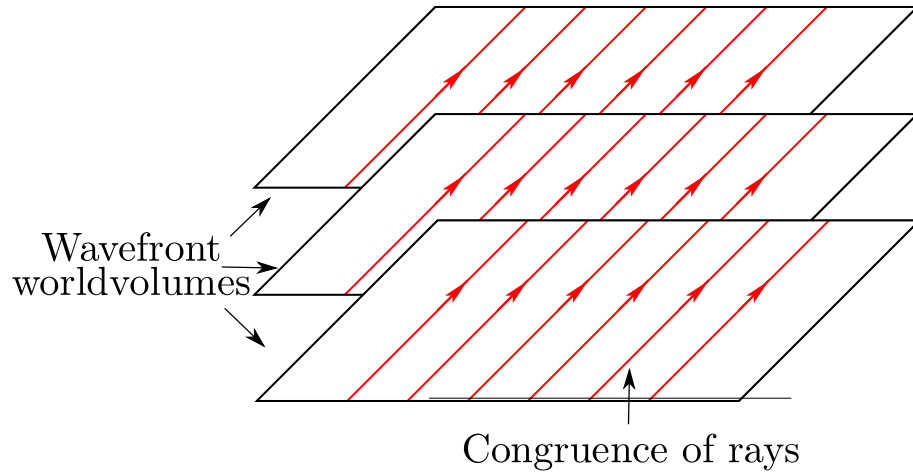


Figure 3: Foliation of a gravitational wave by wavefront worldvolumes

According to the discussion in Section 2.2, each wavefront wordvolume is endowed with a Carrollian structure inherited from the metric structure of the gravitational wave. The congruence of rays defines the gravitational wave via the standard rules of geometric optics. This interpretation is further justified when one considers the following Lemma:

**Lemma 2.35** (cf. [21]). *The wave vector field of any gravitational wave is geodesic for the Levi-Civita connection.*

Furthermore, any gravitational wave admits a class of collinear wave vector fields (i.e. such that  $\xi' = \alpha\xi$ , with  $\alpha \in C^\infty(\mathcal{M})$  a nowhere vanishing function on  $\mathcal{M}$ ), a subclass of which being geodesic<sup>5</sup>.

**Lemma 2.36.** *Let  $(\mathcal{M}, \xi, g)$  be a gravitational wave. The following properties are equivalent:*

1. *The wave vector field  $\xi$  is affine geodesic with respect to the Levi-Civita connection  $\nabla$  associated to  $g$ , i.e.  $\nabla_\xi \xi = 0$ .*
2. *The wave covector field  $\psi$  is invariant, i.e.  $\mathcal{L}_\xi \psi = 0$ .*
3. *The wave vector field  $\xi$  is in the radical of the exterior derivative  $d\psi$  of the wave covector field, i.e.  $d\psi(\xi) = 0$ .*

**Proof:**

Using the metric compatibility of the Levi-Civita connection  $\nabla$ , one can write  $\xi[g(\xi, X)] = g(\nabla_\xi \xi, X) + g(\xi, \nabla_\xi X)$ . In terms of the wave covector field, this reads  $\xi[\psi(X)] = g(\nabla_\xi \xi, X) + \psi(\nabla_\xi X)$ . Reexpressing the second term on the right-hand side using the torsionfree condition  $\nabla_\xi X - \nabla_X \xi = [\xi, X]$ , leads to  $g(\nabla_\xi \xi, X) = \xi[\psi(X)] - \psi(\nabla_X \xi) - \psi([\xi, X])$ . Note that, according to the metric compatibility of  $\nabla$ , one has  $X[g(\xi, \xi)] = 2g(\nabla_X \xi, \xi) = 2\psi(\nabla_X \xi) = 0$ , so that our expression becomes  $g(\nabla_\xi \xi, X) = \xi[\psi(X)] - \psi([\xi, X]) \equiv (\mathcal{L}_\xi \psi)(X)$ , for all  $X \in \Gamma(T\mathcal{M})$ . This proves the equivalence between the properties 1 and 2. The equivalence between the properties 2 and 3 is obtained by making use of Cartan's formula  $(\mathcal{L}_\xi \psi)(X) = d\psi(\xi, X) + X[\psi(\xi)] = d\psi(\xi, X)$ , for all  $X \in \Gamma(T\mathcal{M})$ , since  $\psi(\xi) = 0$ .  $\square$

The next Proposition shows that the most general class of fibered Lorentzian structures whose metric structure projects as an Aristotelian structure on the Platonic

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<sup>5</sup>Note that we use the epithet *geodesic* to designate not-necessarily-affinely-parameterised vector fields, i.e. satisfying  $\nabla_X X = \kappa X$ , with  $\kappa \in C^\infty(\mathcal{M})$  and reserve the term *affine geodesic* to refer to affinely-parameterised vector fields, i.e. satisfying  $\nabla_X X = 0$ .



screen is the important class of *Kundt waves* originally introduced in [49] for entirely different purposes. A detailed account of this class of spacetimes is provided in [50] from which we inherit the following geometric characterisation:

**Definition 2.37** (Kundt wave [50]). *A Kundt wave  $(\mathcal{M}, \xi, g)$  is a fibered Lorentzian structure such that the fundamental vector field is geodesic, expansionless, shearless and twistless.*

Given a Bargmann Basis  $\{\xi, N, e_i\}$ , a Kundt wave is then characterised among fibered Lorentzian structures  $(\mathcal{M}, \xi, g)$  by the conditions:

- The fundamental vector field  $\xi$  is geodesic:

$$\nabla_\xi \xi = \lambda \xi, \text{ with } \lambda \in C^\infty(\mathcal{M}) \quad (2.30)$$

- The three optical scalars vanish:

$$e_i^\mu e_j^\nu \nabla_\mu \psi_\nu = 0. \quad (2.31)$$

Note that the geodesic condition (2.30) ensures that condition (2.31) is independent of the chosen basis (*cf. e.g. [51]*). Before making use of the previous Definition in order to specialise Proposition 2.25, we prove the following technical Lemma:

**Lemma 2.38.** *A fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  induces a projectable ambient Aristotelian structure if and only if the two following conditions are satisfied:*

- $$\psi \wedge d\psi = 0 \quad (2.32)$$

- $$(\mathcal{L}_\xi g)(V, W) = 0, \text{ for all } V, W \in \Gamma(\text{Ker } \psi). \quad (2.33)$$

**Proof:**

The only non-trivial aspect of the proof lies in the implication

$$\psi \wedge d\psi = 0 \Rightarrow \xi \in \text{Rad } \mathcal{L}_\xi g. \quad (2.34)$$

We start by noting that the Frobenius integrability condition ensures the involutivity of the distribution induced by  $\text{Ker } \psi$ , so that

$$d\psi(V, W) = 0 \text{ for all } V, W \in \Gamma(\text{Ker } \psi). \quad (2.35)$$

Now, given a field of observers  $N \in FO(\mathcal{M}, \psi)$  (cf. Definition 2.14 in [5]), any vector field  $X \in \Gamma(T\mathcal{M})$  can be decomposed as  $X = P^N(X) + \psi(X)N$ . Using the previous decomposition, we write:  $d\psi(\xi, X) = d\psi(\xi, P^N(X)) + \psi(X)d\psi(\xi, N) = \psi(X)\mathcal{L}_\xi\psi(N) = 0$  where we used the invariance of the wave covector field (cf. the second Proposition in Lemma 2.36). Using Cartan's formula concludes the proof.  $\square$

**Proposition 2.39.** *A fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  induces a projectable ambient Aristotelian structure if and only if it is a Kundt wave. Therefore, a fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  induces a projectable ambient Augustinian structure if and only if it is a Kundt wave with closed wave covector field  $\psi \equiv g(\xi)$ .*

**Proof:**

We need to show the equivalence between conditions (2.30)-(2.31) and (2.32)-(2.33), starting with the sufficiency. It can be noted that condition (2.31) is preserved by a rescaling of the fundamental vector field  $\xi$ , so that  $\xi$  can be assumed to be affine geodesic (cf. [51]) thus allowing to make use of the following Lemma:

**Lemma 2.40** (cf. e.g. [52], Section 2.4.3). *An affine geodesic vector field is hypersurface-orthogonal if and only if it is twistless.*

According to condition (2.31), the twist  $\omega_{ij} \equiv e_i^\mu e_j^\nu \nabla_\mu \psi_\nu$  vanishes, so that  $\xi$  is hypersurface-orthogonal and condition (2.32) is met. The remainder of the proof will rest on the equality:

$$\nabla_\mu \psi_\nu = \nabla_{(\mu} \psi_{\nu)} + \nabla_{[\mu} \psi_{\nu]} = \partial_{[\mu} \psi_{\nu]} + \frac{1}{2} \mathcal{L}_\xi g_{\mu\nu}. \quad (2.36)$$

Contracting (2.36) with vector fields in  $\Gamma(\text{Ker } \psi)$ , the exterior derivative of  $\psi$  vanishes due to eq.(2.35). If one of these vector fields is the fundamental vector field  $\xi$ , the left-hand side cancels since  $\xi$  is null and affine geodesic. Finally, contracting with  $e_i^\mu$  and  $e_j^\nu$ , the left-hand side vanishes according to eq.(2.31). This concludes the proof that a Kundt wave satisfies conditions (2.32)-(2.33). In order to prove the necessity, we first make use of Lemma 2.35 which ensures that condition (2.30) is satisfied. As for condition (2.31), it follows straightforwardly from eq.(2.36) using condition (2.33) and eq.(2.35).

As noted previously, condition (2.31) is preserved by a rescaling of the fundamental vector field  $\xi$ . Since  $\xi$  is hypersurface-orthogonal, Frobenius theorem ensures that locally there always exists a null vector field  $\bar{\xi} \in \Gamma(T\mathcal{M})$  satisfying  $\xi = \Omega \bar{\xi}$  for some function  $\Omega \in C^\infty(\mathcal{M})$  such that  $\bar{\psi} \equiv g(\bar{\xi}) \in \Omega^1(\mathcal{M})$  is closed<sup>6</sup>. Any Kundt wave then induces a class of projectable ambient Aristotelian structures, a subset of which is Augustinian.  $\square$

We conclude our investigation of gravitational waves by considering the following important subclass:

**Definition 2.41** (Bargmann-Eisenhart wave [22, 20, 21]). *A Bargmann-Eisenhart wave  $\mathcal{B}(\mathcal{M}, \xi, g)$  is a fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  such that the Levi-Civita connection  $\nabla$  associated to  $g$  parallelises the fundamental vector field  $\xi$ . A Bargmann-Eisenhart wave endowed with the associated Levi-Civita connection is called a Bargmann-Eisenhart manifold  $\mathcal{B}(\mathcal{M}, \xi, g, \nabla)$ .*

A fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  is thus a Bargmann-Eisenhart wave if and only if the two following conditions are met:

- $d\psi = 0$
- $\mathcal{L}_\xi g = 0$ .

This set of conditions allows to characterise Bargmann-Eisenhart waves in our terminology as invariant fibered Lorentzian structures inducing a projectable Augustinian structure. The following characterisation of the space of Bargmann-Eisenhart waves follows straightforwardly from Proposition 2.31:

**Proposition 2.42.** *Let  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$  be a projectable ambient Augustinian structure. The set of Bargmann-Eisenhart waves  $(\mathcal{M}, \xi, g)$  admitting  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$  as underlying metric structure forms an affine space canonically isomorphic to the affine space  $\mathcal{P}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  of invariant gravitational potentials.*

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<sup>6</sup>We are grateful to S. Hervik for a useful comment regarding this point.

### 3 Ambient Galilean connections

The previous Section introduced the notion of ambient Leibnizian structures. We discussed in particular the subclasses projecting as a Leibnizian structure on the Platonian screen (projectable ambient Leibnizian structures) and admitting a foliation by Carrollian structures (ambient Aristotelian structures). We then considered how these structures can be extended with a compatible Lorentzian metric, dubbed ambient Lagrangian metric. The next logical step consists in investigating how ambient Leibnizian structures can be endowed with a notion of parallelism, in the guise of a compatible connection and how these connections (called ambient Galilean) induce nonrelativistic Galilean and Carrollian connections. We then discuss the restrictions imposed by the compatibility with a complementing Lagrangian metric.

**Definition 3.1** (Ambient Galilean connections). *An ambient Leibnizian structure supplemented with a compatible Koszul connection is called an ambient Galilean manifold. The compatible Koszul connection is then referred to as an ambient Galilean connection.*

Letting  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be an ambient Leibnizian structure, the compatibility conditions read

1.  $\nabla \xi = 0$
2.  $\nabla \psi = 0$
3.  $\nabla \gamma = 0$ .

These three conditions can be more explicitly stated as:

1.  $\nabla_X \xi = 0$ , for all  $X \in \Gamma(T\mathcal{M})$
2.  $X[\psi(Y)] = \psi(\nabla_X Y)$ , for all  $X, Y \in \Gamma(T\mathcal{M})$
3.  $X[\gamma(V, W)] = \gamma(\nabla_X V, W) + \gamma(V, \nabla_X W)$ , for all  $X \in \Gamma(T\mathcal{M})$  and  $V, W \in \Gamma(\text{Ker } \psi)$ .

Note that the right-hand-side of equation 3. is well defined since  $V \in \Gamma(\text{Ker } \psi)$  implies  $\psi(\nabla_X V) = 0$  (cf. Condition 2.) which in turn, ensures that  $\nabla_X V \in \Gamma(\text{Ker } \psi)$ , for all  $X \in \Gamma(T\mathcal{M})$ .

When the absolute rulers are formulated in terms of a field  $h \in \Gamma(\vee^2(\text{Ann } \xi)^*)$ , Condition 3. can be restated as  $\nabla h = 0$  or equivalently:

$$X[h(\alpha, \beta)] = h(\nabla_X \alpha, \beta) + h(\alpha, \nabla_X \beta), \text{ for all } X \in \Gamma(T\mathcal{M}) \text{ and } \alpha, \beta \in \Gamma(\text{Ann } \xi).$$

Again, we note that the right-hand side is well-defined since the expression  $(\nabla_X \alpha)(\xi) = X[\alpha(\xi)] - \alpha(\nabla_X \xi)$  vanishes for all  $\alpha \in \Gamma(\text{Ann } \xi)$  whenever Condition 1. holds, so that  $\nabla_X \alpha \in \Gamma(\text{Ann } \xi)$ .

### 3.1 Torsionfree connections

**Proposition 3.2.** *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be an ambient Leibnizian structure. Necessary conditions for the existence of a torsionfree ambient Galilean connection compatible with  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  are:*

- $d\psi = 0$
- $\mathcal{L}_\xi \gamma = 0$ .

*In other words,  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  must be a projectable ambient Augustinian structure in order to admit a torsionfree compatible connection.*

These two conditions are reminiscent of the nonrelativistic requirements for the existence of a torsionfree connection compatible respectively with a given Leibnizian structure (cf. Proposition 3.2 in [5]) or Carrollian structure (cf. Proposition A.8). In the following, we will write  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$  for a projectable ambient Augustinian structure and denote  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  the space of torsionfree ambient Galilean connections compatible with  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$ . Since the ambient Augustinian structure  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$  is projectable, it induces a well-defined Augustinian structure  $\bar{\mathcal{S}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  on its Platonic screen. Furthermore, since  $\psi$  is closed, the ambient spacetime  $\mathcal{M}$  admits a foliation by codimension-one hypersurfaces endowed with Carrollian structures. We will denote  $i : \tilde{\mathcal{M}} \hookrightarrow \mathcal{M}$  one of the leaves and  $\tilde{\mathcal{C}}(\tilde{\mathcal{M}}, \tilde{\xi}, \tilde{\gamma})$  the induced Carrollian structure on  $\tilde{\mathcal{M}}$ .

**Proposition 3.3.** *The space  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  of torsionfree ambient Galilean connections possesses the structure of an affine space modelled on the vector space*

$$\mathcal{V}(\mathcal{M}, \xi, \psi, \gamma) \equiv \left\{ S \in \Gamma(\vee^2 T^* \mathcal{M} \otimes T \mathcal{M}) \text{ satisfying conditions a)-c) } \right\}$$

where:

- a)  $S(X, \xi) = 0$  for all  $X \in \Gamma(T \mathcal{M})$
- b)  $\psi(S(X, Y)) = 0$  for all  $X, Y \in \Gamma(T \mathcal{M})$
- c)  $\gamma(S(X, V), W) + \gamma(S(X, W), V) = 0$  for all  $X \in \Gamma(T \mathcal{M})$  and  $V, W \in \Gamma(\text{Ker } \psi)$ .

**Lemma 3.4.** *The vector space  $\mathcal{V}(\mathcal{M}, \xi, \psi, \gamma)$  is isomorphic to the vector space  $\Gamma(\otimes^2 \text{Ann } \xi) \cong \Gamma(\wedge^2 \text{Ann } \xi) \oplus \Gamma(\vee^2 \text{Ann } \xi)$ .*

Explicitly, given an Ehresmann connection  $A \in EC(\mathcal{M}, \xi)$ , one can construct the following non-canonical isomorphism:

$$\begin{aligned} \overset{A}{\varphi} &: \mathcal{V}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \Gamma(\wedge^2 \text{Ann } \xi) \oplus \Gamma(\vee^2 \text{Ann } \xi) \\ &: S_{\mu\nu}^\lambda \mapsto \left( F_{\mu\nu} = -2 \overset{N}{\gamma}_{\lambda[\mu} S_{\nu]\rho}^\lambda N^\rho, \Sigma_{\mu\nu} = A_\lambda S_{\mu\nu}^\lambda \right) \end{aligned} \quad (3.37)$$

whose inverse takes the form

$$\begin{aligned} \overset{A}{\varphi}^{-1} &: \Gamma(\wedge^2 \text{Ann } \xi) \oplus \Gamma(\vee^2 \text{Ann } \xi) \rightarrow \mathcal{V}(\mathcal{M}, \xi, \psi, \gamma) \\ &: (F_{\mu\nu}, \Sigma_{\mu\nu}) \mapsto S_{\mu\nu}^\lambda = \overset{A}{h}^{\lambda\rho} \psi_{(\mu} F_{\nu)\rho} + \xi^\lambda \Sigma_{\mu\nu}. \end{aligned} \quad (3.38)$$

Note that the expression of the 2-form  $F$  is independent of the choice of field of observers  $N$ . In the previous expressions,  $\overset{N}{\gamma}$  (respectively  $\overset{A}{h}$ ) stands for the covariant (respectively contravariant) ambient transverse metric associated to the field of observers  $N$  (respectively the Ehresmann connection  $A$ ).

Explicitly, under a Carroll boost  $A \mapsto A + \alpha$ , with  $\alpha \in \Gamma(\text{Ann } \xi)$  (cf. Appendix A), the isomorphism  $\overset{A}{\varphi}$  transforms as:

$$\overset{A'}{\varphi}(S_{\mu\nu}^\lambda) = \overset{A}{\varphi}(S_{\mu\nu}^\lambda) + (0, \alpha_\lambda S_{\mu\nu}^\lambda) \quad (3.39)$$

$$\overset{A'}{\varphi}^{-1}(F_{\mu\nu}, \Sigma_{\mu\nu}) = \overset{A}{\varphi}^{-1} \left( F_{\mu\nu}, \Sigma_{\mu\nu} - \overset{A}{h}^{\lambda\rho} \alpha_\lambda \psi_{(\mu} F_{\nu)\rho} \right). \quad (3.40)$$

We now construct the following map:

$$\begin{aligned} \Theta &: LP_{\text{inv}}(\mathcal{M}, \xi, \psi) \times \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \Gamma(\wedge^2 \text{Ann } \xi) \oplus \Gamma(\vee^2 \text{Ann } \xi) \\ &: (N, A, \Gamma) \mapsto \left( F_{\mu\nu}^N = -2\gamma_{\lambda[\mu}^N \nabla_{\nu]} N^\lambda, \Sigma_{\mu\nu}^A = -\nabla_{(\mu} A_{\nu)} \right). \end{aligned} \quad (3.41)$$

For all invariant Leibnizian pairs  $(N, A) \in LP_{\text{inv}}(\mathcal{M}, \xi, \psi)$ , the map  $\overset{N,A}{\Theta} \equiv \Theta(N, A, \cdot) : \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \Gamma(\wedge^2 \text{Ann } \xi) \oplus \Gamma(\vee^2 \text{Ann } \xi)$  can be shown to be an affine map modelled on the linear map  $\overset{A}{\varphi}$ , i.e.  $\overset{N,A}{\Theta}(\Gamma') - \overset{N,A}{\Theta}(\Gamma) = \overset{A}{\varphi}(\Gamma' - \Gamma)$ . One can check that the fact that  $(N, A)$  is an invariant Leibnizian pair ensures that  $\overset{N}{F} \in \Gamma(\wedge^2 \text{Ann } \xi)$  and  $\overset{A}{\Sigma} \in \Gamma(\vee^2 \text{Ann } \xi)$ .<sup>7</sup>

Furthermore, Proposition B.4 in [5] ensures that, since  $\overset{N,A}{\Theta}$  is an affine map modelled on the isomorphism of vector spaces  $\overset{A}{\varphi}$ , the kernel of  $\overset{N,A}{\Theta}$  is spanned by a unique  $\overset{N,A}{\Gamma} \in \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$ . We will refer to  $\overset{N,A}{\Gamma} \in \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  as the *torsionfree special ambient connection* associated to the invariant Leibnizian pair  $(N, A)$ . An explicit expression of its components is given by:

$$\overset{N,A}{\Gamma}_{\mu\nu}^\lambda = \xi^\lambda \partial_{(\mu} A_{\nu)} + N^\lambda \partial_{(\mu} \psi_{\nu)} + \frac{1}{2} h^{\lambda\rho} \left[ \partial_\mu \gamma_{\rho\nu}^N + \partial_\nu \gamma_{\rho\mu}^N - \partial_\rho \gamma_{\mu\nu}^N \right].$$

Given an invariant Leibnizian pair  $(N, A)$ , the associated torsionfree special ambient connection can thus be used in order to represent any torsionfree ambient Galilean connection  $\Gamma \in \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  as  $\Gamma = \overset{N,A}{\Gamma} + \overset{A}{\varphi}^{-1} \left( \overset{N,A}{\Theta}(\Gamma) \right)$ . Explicitly, the components of  $\Gamma$  can be written as:

$$\begin{aligned} \Gamma_{\mu\nu}^\lambda &= \xi^\lambda \partial_{(\mu} A_{\nu)} + N^\lambda \partial_{(\mu} \psi_{\nu)} + \frac{1}{2} h^{\lambda\rho} \left[ \partial_\mu \gamma_{\rho\nu}^N + \partial_\nu \gamma_{\rho\mu}^N - \partial_\rho \gamma_{\mu\nu}^N \right] \\ &\quad + h^{\lambda\rho} \psi_{(\mu} \overset{N}{F}_{\nu)\rho} + \xi^\lambda \overset{A}{\Sigma}_{\mu\nu}. \end{aligned} \quad (3.42)$$

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<sup>7</sup>Note that our choice to restrict to *invariant* Leibnizian pairs in the definition of  $\overset{N,A}{\Theta}$  does not constrain the space of torsionfree ambient Galilean connections considered but only the representation we give of such connections. In other words, it only narrows the space of origins of  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  (i.e. the space of torsionfree special ambient connections, cf. below). If one chooses to relax this invariance condition, the map gets modified according to:

$$\overset{N,A}{\Theta} : \Gamma \mapsto \left( F_{\mu\nu}^N = -2\gamma_{\lambda[\mu}^N \nabla_{\nu]} N^\lambda + 2\gamma_{\lambda[\mu}^N A_{\nu]} \mathcal{L}_\xi N^\lambda, \Sigma_{\mu\nu}^A = -\nabla_{(\mu} A_{\nu)} + A_{(\mu} \mathcal{L}_\xi A_{\nu)} \right).$$

This is the most general expression of a torsionfree ambient Galilean connection. The arbitrariness is encoded in  $\overset{N}{F}$  and  $\overset{A}{\Sigma}$  corresponding respectively to the arbitrariness in a torsionfree Galilean connection (*cf.* Section 3 in [5]) and a torsionfree Carrollian connection (*cf.* Section A.2).

The superscript acts here as a reminder of the fact that  $\overset{N,A}{\Theta}$  is not canonical. Under a change of Leibnizian pair (2.16), this map varies according to

$$\overset{N',A'}{\Theta}(\Gamma) = \overset{N,A}{\Theta}(\Gamma) + \left( 2\partial_{[\mu} \overset{N,V}{\dot{\Phi}}_{\nu]}, -\frac{1}{2}\mathcal{L}_{\alpha^b} \overset{N}{\gamma}_{\mu\nu} + \overset{A}{h}^{\lambda\rho} \alpha_\lambda \psi_{(\mu} \overset{N}{F}_{\nu)\rho} + \partial_{(\mu} (a + \alpha(V)) \psi_{\nu)} \right)$$

where  $\alpha^b \equiv \overset{A}{h}(\alpha) \in \Gamma(T\mathcal{M})$ . This expression can then be used to define an action of the group  $\mathcal{H}_{\text{inv}}(d)$  on the space  $\mathcal{R}(\mathcal{M}, \xi, \psi, \gamma) \equiv LP_{\text{inv}}(\mathcal{M}, \xi, \psi) \times (\Gamma(\wedge^2 \text{Ann } \xi) \oplus \Gamma(\vee^2 \text{Ann } \xi))$  as:

$$\begin{cases} N' = N + P^A(V) + a\xi \\ A' = A + \bar{P}^N(\alpha) - (a + \alpha(V))\psi \\ F' = F + d\overset{N,V}{\dot{\Phi}} \\ \Sigma'_{\mu\nu} = \Sigma_{\mu\nu} - \frac{1}{2}\mathcal{L}_{\alpha^b} \overset{N}{\gamma}_{\mu\nu} + \overset{A}{h}^{\lambda\rho} \alpha_\lambda \psi_{(\mu} F_{\nu)\rho} + \partial_{(\mu} (a + \alpha(V)) \psi_{\nu)}. \end{cases} \quad (3.43)$$

Projecting the transformation law of the couple  $(N, F)$  on the Platonic screen  $\bar{\mathcal{M}}$  leads to the group action of the nonrelativistic Milne group  $\Gamma(\text{Ker } \bar{\psi})$  on  $FO(\bar{\mathcal{M}}, \bar{\psi}) \times \Omega^2(\bar{\mathcal{M}})$  (*cf.* eq. (3.32) in [5]):

$$(\bar{N}, \bar{F}) \mapsto (\bar{N}', \bar{F}') = \left( \bar{N} + \bar{V}, \bar{F} + d\overset{\bar{N}, \bar{V}}{\dot{\Phi}} \right). \quad (3.44)$$

where  $\bar{N} \equiv \pi_* N$ ,  $\pi^* \bar{F} \equiv F$  and  $\bar{V} \equiv \pi_* V$  and  $\overset{\bar{N}, \bar{V}}{\dot{\Phi}} \equiv \bar{\gamma}(\bar{V}) - \frac{1}{2}\bar{\gamma}(\bar{V}, \bar{V})\bar{\psi}$ .

Similarly, one can recover the group action of  $\Gamma(\text{Ann } \tilde{\xi})$  on  $EC(\tilde{\mathcal{M}}, \tilde{\xi}) \times \Gamma(\vee^2 \text{Ann } \tilde{\xi})$  (*cf.* eq.(A.62)) by pullback of the transformation relations of the couple  $(A, \Sigma)$  on a leaf  $i : \tilde{\mathcal{M}} \hookrightarrow \mathcal{M}$ :

$$(\tilde{A}, \tilde{\Sigma}) \mapsto \left( \tilde{A} + \tilde{\alpha}, \tilde{\Sigma} - \frac{1}{2}\mathcal{L}_{\tilde{\alpha}^b} \tilde{\gamma} \right) \quad (3.45)$$



where  $\tilde{A} \equiv i^* A$ ,  $\tilde{\Sigma} \equiv i^* \Sigma$ ,  $\tilde{\alpha} \equiv i^* \alpha$  and  $\tilde{\alpha}^b \equiv \tilde{h}^{\tilde{A}}(\tilde{\alpha})$ .

**Definition 3.5** (Ambient gravitational fieldstrength). *An  $\mathcal{H}_{\text{inv}}(d)$ -orbit in  $\mathcal{R}(\mathcal{M}, \xi, \psi, \gamma)$  is dubbed an ambient gravitational fieldstrength. The space of ambient gravitational fieldstrengths will be denoted  $\mathcal{F}(\mathcal{M}, \xi, \psi, \gamma) \equiv \mathcal{R}(\mathcal{M}, \xi, \psi, \gamma) / \mathcal{H}_{\text{inv}}(d)$ .*

Using this terminology, we can further characterise the affine space of torsionfree ambient Galilean connections. The following Proposition is a straightforward application of Proposition B.4 in [5]:

**Proposition 3.6.** *The space  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  of torsionfree ambient Galilean connections compatible with a given ambient Augustinian structure  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$  possesses the structure of an affine space canonically isomorphic to the affine space  $\mathcal{F}(\mathcal{M}, \xi, \psi, \gamma)$  of ambient gravitational fieldstrengths.*

This characterisation of torsionfree ambient Galilean connections in terms of ambient gravitational fieldstrengths mimics the construction of Proposition 3.14 in [5]. With the help of these two characterisations, we establish the following fact:

**Corollary 3.7.** *Let  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$  be a projectable ambient Augustinian structure and denote  $\bar{\mathcal{S}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  the induced Augustinian structure on the Platonic screen  $\bar{\mathcal{M}}$ . Let  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  and  $\bar{\mathcal{D}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  denote the affine spaces of torsionfree Galilean connections compatible with  $\mathcal{S}$  and  $\bar{\mathcal{S}}$ , respectively. There is a surjective affine map  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \bar{\mathcal{D}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$ .*

The proof takes advantage of the characterisation of (ambient) torsionfree Galilean connections in terms of (ambient) gravitational fieldstrengths, allowing to construct the map:

$$\mathcal{F}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \bar{\mathcal{F}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma}) : [N, A, F, \Sigma] \mapsto [\bar{N}, \bar{F}]$$

which is obviously well-defined (cf. eq.(3.44)) and surjective. In other words, any torsionfree Galilean manifold can be obtained as projection of a (class of) ambient torsionfree Galilean manifolds. Componentwise, the projection of (3.42) takes the form:

$$\bar{\Gamma}_{\bar{\mu}\bar{\nu}}^{\bar{\lambda}} = \bar{N}^{\bar{\lambda}} \partial_{(\bar{\mu}} \bar{\psi}_{\bar{\nu})} + \frac{1}{2} \bar{h}^{\bar{\lambda}\bar{\rho}} \left[ \partial_{\bar{\mu}} \bar{\gamma}_{\bar{\rho}\bar{\nu}}^{\bar{N}} + \partial_{\bar{\nu}} \bar{\gamma}_{\bar{\rho}\bar{\mu}}^{\bar{N}} - \partial_{\bar{\rho}} \bar{\gamma}_{\bar{\mu}\bar{\nu}}^{\bar{N}} \right] + \bar{h}^{\bar{\lambda}\bar{\rho}} \bar{\psi}_{(\bar{\mu}} \bar{F}_{\bar{\nu})\bar{\rho}}^{\bar{N}}.$$

The previous Corollary can be dualised to account for the embedding of Carrollian manifolds inside ambient Galilean manifolds:

**Proposition 3.8.** *Let  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$  be a projectable ambient Augustinian structure and denote  $i : \tilde{\mathcal{M}} \hookrightarrow \mathcal{M}$  one of the leaves foliating  $\mathcal{M}$  and  $\tilde{\mathcal{C}}(\tilde{\mathcal{M}}, \tilde{\xi}, \tilde{\gamma})$  the induced Carrollian structure. Let  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  (resp.  $\tilde{\mathcal{D}}(\tilde{\mathcal{M}}, \tilde{\xi}, \tilde{\gamma})$ ) denote the affine spaces of torsionfree ambient Galilean connections (resp. torsionfree Carrollian connections) compatible with  $\mathcal{S}$  (resp.  $\tilde{\mathcal{C}}$ ). There is a surjective affine map  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \tilde{\mathcal{D}}(\tilde{\mathcal{M}}, \tilde{\xi}, \tilde{\gamma})$ .*

The surjective map is explicitly given by:

$$\mathcal{F}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \tilde{\mathcal{F}}(\tilde{\mathcal{M}}, \tilde{\xi}, \tilde{\gamma}) : [N, A, F, \Sigma] \mapsto [\tilde{A}, \tilde{\Sigma}]$$

This map in turn induces a surjective map between the affine spaces of torsion-free ambient Galilean connections and torsionfree Carrollian connections which, in component, reads:

$$\begin{aligned} \Gamma_{\mu\nu}^\lambda &= \xi^\lambda \partial_{(\mu} A_{\nu)} + N^\lambda \partial_{(\mu} \psi_{\nu)} + \frac{1}{2} \tilde{h}^{\lambda\rho} \left[ \partial_\mu \tilde{\gamma}_{\rho\nu}^N + \partial_\nu \tilde{\gamma}_{\rho\mu}^N - \partial_\rho \tilde{\gamma}_{\mu\nu}^N \right] + \tilde{h}^{\lambda\rho} \psi_{(\mu} \tilde{F}_{\nu)\rho}^N + \xi^\lambda \tilde{\Sigma}_{\mu\nu}^A \\ &\quad \downarrow \\ \tilde{\Gamma}_{\tilde{\mu}\tilde{\nu}}^{\tilde{\lambda}} &= \tilde{\xi}^{\tilde{\lambda}} \partial_{(\tilde{\mu}} \tilde{A}_{\tilde{\nu})} + \frac{1}{2} \tilde{h}^{\tilde{\lambda}\tilde{\rho}} \left[ \partial_{\tilde{\mu}} \tilde{\gamma}_{\tilde{\rho}\tilde{\nu}} + \partial_{\tilde{\nu}} \tilde{\gamma}_{\tilde{\rho}\tilde{\mu}} - \partial_{\tilde{\rho}} \tilde{\gamma}_{\tilde{\mu}\tilde{\nu}} \right] + \tilde{\xi}^{\tilde{\lambda}} \tilde{\Sigma}_{\tilde{\mu}\tilde{\nu}}^{\tilde{A}}. \end{aligned}$$

## 3.2 Torsional connections

We now adress the issue of torsional ambient Galilean connections by mimicking the previous discussion.

**Proposition 3.9.** *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be an ambient Leibnizian structure. The torsion tensor  $T \in \Gamma(\wedge^2 T^* \mathcal{M} \otimes T \mathcal{M})$  of a torsional ambient Galilean connection compatible with  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  must satisfy the two following relations:*

1.  $\psi(T(X, Y)) = d\psi(X, Y)$  for all  $X, Y \in \Gamma(T \mathcal{M})$ ,
2.  $\mathcal{L}_\xi \tilde{\gamma}^N(V, W) = \tilde{\gamma}^N(V, T(\xi, W)) + \tilde{\gamma}^N(W, T(\xi, V))$  for all  $V, W \in \Gamma(\text{Ker } \psi / \text{Span } \xi)$  and  $N \in FO(\mathcal{M}, \psi)$ .

A torsion tensor satisfying these two relations will be said compatible with the ambient Leibnizian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$ .

Whenever the ambient absolute clock  $\psi$  is projectable (i.e.  $\mathcal{L}_\xi \psi = 0$ ), the second condition is equivalent to  $\mathcal{L}_\xi \gamma(V, W) = \gamma(V, T(\xi, W)) + \gamma(W, T(\xi, V))$  for all  $V, W \in \Gamma(\text{Ker } \psi / \text{Span } \xi)$ .

The two previous conditions constrain  $\frac{(d+1)d}{2} + \frac{(d+2)(d+1)}{2} = (d+1)^2$  components of the torsion. This fact provides an *a posteriori* justification regarding the appearance of the tensors  $F \in \Gamma(\wedge^2 \text{Ann } \xi)$  and  $\Sigma \in \Gamma(\vee^2 \text{Ann } \xi)$  since the fibers of the vector bundle  $(\wedge^2 \text{Ann } \xi \oplus \vee^2 \text{Ann } \xi)$  have dimension  $\frac{d(d+1)}{2} + \frac{(d+1)(d+2)}{2} = (d+1)^2$ . The amount of arbitrariness in the choice of a (potentially torsional) compatible connection is thus the same for ambient Leibnizian structures than for Lorentzian structures.

Since we are mostly interested with ambient structures admitting a well-defined projection on their Platonic screen, we restrict the scope of the analysis to projectable ambient Leibnizian structures  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$ . A torsion tensor  $T \in \Gamma(\wedge^2 T^* \mathcal{M} \otimes T\mathcal{M})$  will be said to be *projectable* if the following relations hold:

- $\mathcal{L}_\xi T = 0$
- $T(\xi, X) = 0$  for all  $X \in \Gamma(T\mathcal{M})$ .

Given a projectable ambient Leibnizian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$ , we denote  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  the set of compatible ambient Galilean connections with projectable torsions<sup>8</sup>. An ambient Galilean manifold  $\mathcal{G}(\mathcal{M}, \xi, \psi, \gamma, \nabla)$  will be said projectable if both the underlying Leibnizian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  and torsion tensor associated to  $\nabla$  are projectable.

**Proposition 3.10.** *The space  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  possesses the structure of an affine space modelled on the vector space  $\mathcal{V}(\mathcal{M}, \xi, \psi, \gamma)$  defined as:*

$$\mathcal{V}(\mathcal{M}, \xi, \psi, \gamma) \equiv \left\{ S \in \Gamma_{\text{inv}}(T^* \mathcal{M} \otimes T^* \mathcal{M} \otimes T\mathcal{M}) \text{ satisfying conditions a)-d)} \right\}$$

where:

$$a) \quad S(X, \xi) = 0 \text{ for all } X \in \Gamma(T\mathcal{M})$$

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<sup>8</sup>We pursue with the notation convention used in [5] and make use of the same symbols for the various spaces and maps encountered in the torsionfree and torsional cases in order to emphasise the similitude in the logic of the arguments.

- b)  $S(\xi, X) = 0$  for all  $X \in \Gamma(T\mathcal{M})$
- c)  $\psi(S(X, Y)) = 0$  for all  $X, Y \in \Gamma(T\mathcal{M})$
- d)  $\gamma(S(X, V), W) + \gamma(S(X, W), V) = 0$  for all  $X \in \Gamma(T\mathcal{M})$  and  $V, W \in \Gamma(\text{Ker } \psi)$ .

**Lemma 3.11.** *The vector space  $\mathcal{V}(\mathcal{M}, \xi, \psi, \gamma)$  is isomorphic to the vector space  $\mathcal{W}(\mathcal{M}, \xi, \psi, \gamma) \equiv \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi) \oplus \Gamma_{\text{inv}}(\otimes^2 \text{Ann } \xi) \oplus \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi \otimes (\text{Ker } \psi / \text{Span } \xi))$ .*

Explicitly, given an invariant Leibnizian pair  $(N, A) \in LP_{\text{inv}}(\mathcal{M}, \xi, \psi)$ , one can construct the following non-canonical isomorphism:

$$\begin{aligned} \overset{N,A}{\varphi} &: \mathcal{V}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \mathcal{W}(\mathcal{M}, \xi, \psi, \gamma) \\ &: S_{\mu\nu}^\lambda \mapsto \left( F_{\mu\nu} = -2\gamma_{\lambda[\mu}^N S_{\nu]\rho}^\lambda N^\rho, \Sigma_{\mu\nu} = A_\lambda S_{\mu\nu}^\lambda, [U_{\mu\nu}^\lambda] = [S_{[\mu\nu]}^\lambda] \right) \end{aligned} \quad (3.46)$$

whose inverse takes the form

$$\begin{aligned} \overset{N,A}{\varphi}^{-1} &: \mathcal{W}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \mathcal{V}(\mathcal{M}, \xi, \psi, \gamma) \\ &: (F_{\mu\nu}, \Sigma_{\mu\nu}, [U_{\mu\nu}^\lambda]) \mapsto S_{\mu\nu}^\lambda = \overset{A}{h}^{\lambda\rho} \psi_{(\mu} F_{\nu)\rho} + \xi^\lambda \Sigma_{\mu\nu} + P^A (U_{\mu\nu}^\lambda) + 2 \overset{A}{h}^{\sigma\lambda} U_{\sigma(\mu}^\rho \gamma_{\nu)\rho}^N. \end{aligned} \quad (3.47)$$

where  $U_{\mu\nu}^\lambda \in \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi \otimes (\text{Ker } \psi))$  is an arbitrary representative of the equivalence class  $[U_{\mu\nu}^\lambda] \in \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi \otimes (\text{Ker } \psi / \text{Span } \xi))$ .

We now construct the following map:

$$\begin{aligned} \Theta &: LP_{\text{inv}}(\mathcal{M}, \xi, \psi) \times \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \mathcal{W}(\mathcal{M}, \xi, \psi, \gamma) \\ &: (N, A, \Gamma) \mapsto \left( F_{\mu\nu}^N = -2\gamma_{\lambda[\mu}^N \nabla_{\nu]} N^\lambda, \overset{A}{\Sigma}_{\mu\nu} = -\nabla_\mu A_\nu, [U_{\mu\nu}^N] = [P^N(\Gamma_{[\mu\nu]}^\lambda)] \right). \end{aligned}$$

For all invariant Leibnizian pair  $(N, A) \in LP_{\text{inv}}(\mathcal{M}, \xi, \psi)$ , the map  $\overset{N,A}{\Theta} \equiv \Theta((N, A), \cdot) : \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \mathcal{W}(\mathcal{M}, \xi, \psi, \gamma)$  can be shown to be an affine map modelled on the linear map  $\overset{N,A}{\varphi}$  i.e.  $\overset{N,A}{\Theta}(\Gamma') - \overset{N,A}{\Theta}(\Gamma) = \overset{N,A}{\varphi}(\Gamma' - \Gamma)$ .

Given an invariant Leibnizian pair  $(N, A) \in LP_{\text{inv}}(\mathcal{M}, \xi, \psi)$  and an ambient Galilean connection  $\Gamma \in \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$ , we call the couple  $\overset{N,A}{\Theta}(\Gamma)$  the *ambient torsional gravitational fieldstrength measured by  $(N, A)$* . This piece of terminology as well as the fact that  $\overset{N,A}{\Theta}$  is an affine map modelled on the isomorphism of vector spaces  $\overset{N,A}{\varphi}$  allows us to formulate the following Proposition:

**Proposition 3.12** (Torsional special ambient connection). *Given an invariant Leibnizian pair  $(N, A) \in LP_{\text{inv}}(\mathcal{M}, \xi, \psi)$ , there is a unique ambient Galilean connection  $\overset{N,A}{\Gamma} \in \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  compatible with the ambient Leibnizian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  such that the torsional ambient gravitational fieldstrength measured by  $(N, A)$  with respect to  $\overset{N,A}{\Gamma}$  vanishes. It will be called  $\overset{N,A}{\Gamma}$  the torsional special ambient connection associated to  $(N, A)$ .*

In other words,  $\text{Ker } \overset{N,A}{\Theta} \cong \text{Span} \left\{ \overset{N,A}{\Gamma} \right\}$ . An explicit expression for the components of  $\overset{N,A}{\Gamma}$  is given by:

$$\overset{N,A}{\Gamma}_{\mu\nu}^{\lambda} = \xi^{\lambda} \partial_{\mu} A_{\nu} + N^{\lambda} \partial_{\mu} \psi_{\nu} + \frac{1}{2} h^{\lambda\rho} \left[ \partial_{\mu} \overset{N}{\gamma}_{\rho\nu} + \partial_{\nu} \overset{N}{\gamma}_{\rho\mu} - \partial_{\rho} \overset{N}{\gamma}_{\mu\nu} \right]. \quad (3.48)$$

Given an invariant Leibnizian pair  $(N, A)$ , the associated torsional special ambient connection can thus be used in order to represent any Galilean connection  $\Gamma \in \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  as  $\Gamma = \overset{N,A}{\Gamma} + \overset{N,A}{\varphi}^{-1} \left( \overset{N,A}{\Theta}(\Gamma) \right)$ . Explicitly, the components of  $\Gamma$  can be written as:

$$\begin{aligned} \Gamma_{\mu\nu}^{\lambda} &= \xi^{\lambda} \partial_{\mu} A_{\nu} + N^{\lambda} \partial_{\mu} \psi_{\nu} + \frac{1}{2} h^{\lambda\rho} \left[ \partial_{\mu} \overset{N}{\gamma}_{\rho\nu} + \partial_{\nu} \overset{N}{\gamma}_{\rho\mu} - \partial_{\rho} \overset{N}{\gamma}_{\mu\nu} \right] + h^{\lambda\rho} \psi_{(\mu} \overset{N}{F}_{\nu)\rho} + \xi^{\lambda} \overset{A}{\Sigma}_{\mu\nu} \\ &\quad + P^A \left( \overset{N}{U}_{\mu\nu}^{\lambda} \right) + 2 h^{\sigma\lambda} \overset{A}{U}_{\sigma(\mu}^{\rho} \overset{N}{\gamma}_{\nu)\rho}. \end{aligned} \quad (3.49)$$

We now define the following action of the group  $\mathcal{H}_{\text{inv}}(d)$  on the space  $\mathcal{R}(\mathcal{M}, \xi, \psi, \gamma) \equiv LP_{\text{inv}}(\mathcal{M}, \xi, \psi) \times \mathcal{W}(\mathcal{M}, \xi, \psi, \gamma)$ :

$$\begin{cases} F'_{\mu\nu} = F_{\mu\nu} + 2\partial_{[\mu} \overset{N,V}{\Phi}_{\nu]} - 2\overset{N}{\gamma}_{\alpha\lambda} V^{\alpha} U_{\mu\nu}^{\lambda} + \gamma(V, V) \partial_{[\mu} \psi_{\nu]} \\ \Sigma'_{\mu\nu} = \Sigma_{\mu\nu} - \frac{1}{2} \mathcal{L}_{\alpha^b} \overset{N}{\gamma}_{\mu\nu} + h^{\lambda\rho} \alpha_{\lambda} \psi_{(\mu} F_{\nu)\rho} + \partial_{\mu} \left( a + \alpha(V) \right) \psi_{\nu} + 2h^{\sigma\lambda} \alpha_{\lambda} U_{\sigma(\mu}^{\rho} \overset{N}{\gamma}_{\nu)\rho} \\ [U'^{\lambda}_{\mu\nu}] = [U^{\lambda}_{\mu\nu} - V^{\lambda} \partial_{[\mu} \psi_{\nu]}] \end{cases} \quad (3.50)$$

and the usual transformation law of the Leibnizian pair.

**Definition 3.13** (Torsional ambient gravitational fieldstrength). *An  $\mathcal{H}_{\text{inv}}(d)$ -orbit in  $\mathcal{R}(\mathcal{M}, \xi, \psi, \gamma)$  is dubbed a torsional ambient gravitational fieldstrength. The space of torsional ambient gravitational fieldstrengths will be denoted  $\mathcal{F}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma) \equiv \mathcal{R}(\mathcal{M}, \xi, \psi, \gamma) / \mathcal{H}_{\text{inv}}(d)$ .*

Using this terminology, we can further characterise the affine space of ambient

Galilean connections as follows:

**Proposition 3.14.** *The space  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  of ambient Galilean connections with projectable torsion compatible with a given ambient Leibnizian structure possesses the structure of an affine space canonically isomorphic to the affine space  $\mathcal{F}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  of torsional ambient gravitational fieldstrengths.*

The next corollary follows straightforwardly from Proposition 2.15 using the characterisation of torsional (ambient) Galilean connections in terms of torsional (ambient) gravitational fieldstrengths (cf. Proposition 3.14 and Proposition 4.10 in [5]):

**Corollary 3.15.** *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be a projectable ambient Leibnizian structure and denote  $\bar{\mathcal{L}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  the induced Leibnizian structure on the Platonic screen  $\bar{\mathcal{M}}$ . Let  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  and  $\bar{\mathcal{D}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  denote the affine spaces of torsional Galilean connections compatible with  $\mathcal{L}$  and  $\bar{\mathcal{L}}$ , respectively. There is a surjective affine map  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \bar{\mathcal{D}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$ .*

The surjective map is explicitly given by:

$$\mathcal{F}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \bar{\mathcal{F}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma}) : [N, A, F, \Sigma, [U]] \mapsto [\bar{N}, \bar{F}, \bar{U}]$$

where  $\bar{N} \equiv \pi_* N$ ,  $\pi^* \bar{F} \equiv F$  and  $\bar{U} \equiv \pi^* U$ , with  $U$  a representative of  $[U]$ .

### 3.3 Projectable Lorentzian manifolds

We conclude our study of ambient connections by investigating the interplay between parallelism compatible with fibered Lorentzian structures and ambient Leibnizian structures.

**Proposition 3.16.** *Let  $(\mathcal{M}, \xi, g)$  be a fibered Lorentzian structure and let  $\nabla$  be a Koszul connection compatible with  $g$ . The Koszul connection  $\nabla$  parallelises  $\xi$  if and only if the torsion tensor  $T \in \Gamma(\wedge^2 T^* \mathcal{M} \otimes T \mathcal{M})$  associated to  $\nabla$  satisfy the two following relations:*

1.  $\psi(T(X, Y)) = d\psi(X, Y)$  for all  $X, Y \in \Gamma(T \mathcal{M})$
2.  $\mathcal{L}_\xi g(X, Y) = g(X, T(\xi, Y)) + g(Y, T(\xi, X))$  for all  $X, Y \in \Gamma(T \mathcal{M})$ .

A torsion tensor satisfying these two conditions will be said compatible with  $(\mathcal{M}, \xi, g)$ .

Assuming that  $T$  is projectable, equation 2. ensures that  $\xi$  is Killing with respect to  $g$ . This motivates the introduction of the following class of manifolds:

**Definition 3.17** (Projectable Lorentzian manifold). *An invariant fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  supplemented with a compatible connection  $\nabla$  with projectable torsion is called a projectable Lorentzian manifold  $(\mathcal{M}, \xi, g, \nabla)$ .*

We start by noting that the class of *torsionfree* projectable Lorentzian manifolds identifies with the class of Bargmann-Eisenhart manifolds (*cf.* Definition 2.41).

Obviously, a Koszul connection compatible with a given fibered Lorentzian structure is an ambient Galilean connection for the induced Leibnizian structure since:

$$\begin{cases} \nabla \xi = 0 \\ \nabla g = 0 \end{cases} \Rightarrow \begin{cases} \nabla \xi = 0 \\ \nabla \psi = 0 \\ \nabla \gamma = 0. \end{cases}$$

**Proposition 3.18.** *Any projectable Lorentzian manifold  $(\mathcal{M}, \xi, g, \nabla)$  defines a projectable ambient Galilean manifold  $(\mathcal{M}, \xi, \psi, \gamma, \nabla)$  with  $\psi \equiv g(\xi)$  and  $\gamma$  the restriction of  $g$  to  $\text{Ker } \psi$ .*

However, the converse is not true: only a subset of projectable ambient Galilean manifolds can be complemented by a compatible projectable Lorentzian manifold. The following Lemma displays the corresponding restrictions:

**Lemma 3.19.** *Let  $(\mathcal{M}, \xi, \psi, \gamma, \nabla)$  be a projectable ambient Galilean manifold and let  $[N, A] \in \mathcal{P}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  be an invariant ambient gravitational potential. Let  $g \in \mathcal{L}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  denote the invariant ambient Lagrangian metric associated to  $[N, A]$ . Given an invariant Leibnizian pair  $(N, A) \in [N, A]$ , let us call  $\overset{N, A}{\Theta}(\nabla) = (F, \Sigma, [U])$ . The ambient Galilean connection  $\nabla$  is compatible with  $g$  if and only if*

- $\Sigma_{[\mu\nu]} = -\frac{1}{2}F_{\mu\nu}$
- $\Sigma_{(\mu\nu)} = -\frac{1}{2}\mathcal{L}_N g_{\mu\nu} - N^\rho g_{\sigma(\mu} T_{\nu)\rho}^\sigma$

where  $T \in \Gamma(\wedge^2 T^* \mathcal{M} \otimes T \mathcal{M})$  stands for the torsion of  $\nabla$ .

Let us first focus on the torsionfree case for which, given an invariant Leibnizian pair  $(N, A)$ , the arbitrariness in the ambient Galilean connection is encoded in  $F \in$

$\Gamma(\wedge^2 \text{Ann } \xi)$  and  $\Sigma \in \Gamma(\vee^2 \text{Ann } \xi)$ . The compatibility condition with the Lagrangian metric  $g$  ensures that  $\nabla$  is the associated Levi-Civita connection, which is thus determined univoquely by  $[N, A]$ . On the one hand, the second equality of Lemma 3.19 reads in the torsionfree case as  $\Sigma = -\frac{1}{2}\mathcal{L}_N g$ . In other words, the pullback of  $\Sigma$  on a given wavefront worldvolume  $i : \tilde{\mathcal{M}} \hookrightarrow \mathcal{M}$  of a Bargmann-Eisenhart wave identifies with the transverse extrinsic curvature on  $\tilde{\mathcal{M}}$  (cf. [44] for terminology). On the other hand, the torsionfree condition ensures that  $\Sigma_{[\mu\nu]} \equiv -\nabla_{[\mu} A_{\nu]} = -\partial_{[\mu} A_{\nu]}$  so that the first equality reduces to  $F = dA$ . The 2-form  $F$  takes then the interpretation of the curvature of the principal  $\mathbb{R}$ -connection  $A \in PC(\mathcal{M}, \xi)$ . We hence recover the result of [22] stating that the Galilean connection obtained by projection of the Levi-Civita connection associated to a Bargmann-Eisenhart wave is *Newtonian* (i.e. with closed 2-form  $\bar{F}$ ). More precisely, one can formulate the following Proposition:

**Proposition 3.20** (cf. [22]). *Let  $\pi : \mathcal{M} \rightarrow \bar{\mathcal{M}}$  be an ambient structure. Bargmann-Eisenhart manifolds on  $\mathcal{M}$  are in bijective correspondence with torsionfree Newtonian manifolds on the Platonic screen  $\bar{\mathcal{M}}$ .*

The following diagram schematically wraps up the embedding procedure in the torsionfree case:

$$\begin{array}{ccc}
 \mathcal{S}(\mathcal{M}, \xi, \psi, \gamma) & \xrightarrow{[N, A]} & \mathcal{B}(\mathcal{M}, \xi, g, \nabla) \\
 \downarrow \pi & & \downarrow \pi \\
 \bar{\mathcal{S}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma}) & \xrightarrow{[\bar{N}, [\bar{A}]]} & \bar{\mathcal{N}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma}, \bar{\nabla})
 \end{array}$$

The upper row describes the definition of a Bargmann-Eisenhart manifold  $\mathcal{B}(\mathcal{M}, \xi, g, \nabla)$  from a projectable ambient Augustinian structure  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$  by means of an invariant ambient gravitational potential  $[N, A] \in \mathcal{P}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  (cf. Proposition 2.42). Similarly, on the Platonic screen  $\bar{\mathcal{M}}$ , the Newtonian manifold  $\bar{\mathcal{N}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma}, \bar{\nabla})$  is obtained by adjunction of a gravitational potential  $[\bar{N}, [\bar{A}]] \in \bar{\mathcal{P}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  to the Augustinian structure  $\bar{\mathcal{S}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  (cf. Proposition 3.20 in [5]). The vertical left map from the space of projectable Augustinian structures on  $\mathcal{M}$  to the space of Augustinian structures on the Platonic screen  $\bar{\mathcal{M}}$  has been shown to be biject-



tive in Proposition 2.15. The isomorphism of affine spaces between the spaces of (ambient) gravitational potentials has been established in Proposition 2.29, thus ensuring that the vertical right map defines an isomorphism between the affine spaces of Bargmann-Eisenhart manifolds compatible with  $\mathcal{S}(\mathcal{M}, \xi, \psi, \gamma)$  and Newtonian manifolds compatible with  $\bar{\mathcal{S}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$ . This correspondence can also be appreciated using Lagrangian metrics (cf. Proposition 2.32).

The relativistic spacetimes admitting a compatible torsionfree special ambient connections constitute a physically important<sup>9</sup> subclass of Bargmann-Eisenhart waves since, as detailed in the following corollary, they possess two commuting light-like Killing vector fields:

**Corollary 3.21.** *Let  $\mathcal{B}(\mathcal{M}, \xi, g, \nabla)$  be a Bargmann-Eisenhart manifold with absolute clock  $\psi$  and  $N \in FLO_{\text{inv}}(\mathcal{M}, \psi, g)$  be an invariant field of light-like observers. The Levi-Civita connection  $\Gamma \in \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  is the torsionfree special ambient connection associated to the invariant Leibnizian pair  $(N, A) \in LP(\mathcal{M}, \xi, \psi)$  (with  $A \equiv g(N)$ ) if and only if*

1.  $\xi$  and  $N$  are two commuting null Killing vector fields for the Bargmann-Eisenhart metric  $g$ ,
2.  $\psi$  and  $A$  are two flat (i.e.  $d\psi = 0 = dA$ ) principal  $\mathbb{R}$ -connections (i.e.  $\psi \in PC(\mathcal{M}, N)$  and  $A \in PC(\mathcal{M}, \xi)$ ).

The first point implies that  $\xi$  and  $N$  provide a basis of fundamental vector fields on the principal  $\mathbb{R}^2$ -bundle  $(\mathcal{M}, \xi, N)$ . The second point is equivalent to the fact that  $\psi \oplus A$  is a flat principal  $\mathbb{R}^2$ -connection on the previous bundle.

We now move away from the torsionfree case and propose the following generalisation of Proposition 3.20:

**Proposition 3.22.** *Let  $\pi : \mathcal{M} \rightarrow \bar{\mathcal{M}}$  be an ambient structure. Projectable Lorentzian manifolds on  $\mathcal{M}$  are in bijective correspondence with Lagrangian manifolds on the Platonic screen  $\bar{\mathcal{M}}$ .*

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<sup>9</sup>In the spacelike case, the analogous subclass is of great interest for integrability and duality issues in general relativity. The same should be true in the light-like case (cf. the introductive comments in [53]).

Lagrangian manifolds (*i.e.* a Leibnizian structure endowed with a compatible Galilean connection and a class of Lagrangian metrics) were introduced in Definition 3.30 of [5]. We start with a heuristic remark by noting that, taking into account the projectability condition ( $T(\xi, \cdot) = 0$ ) as well as the constraint on the “timelike part” ( $\psi(T) = d\psi$ ), the number of free components of a torsion tensor on the  $(d+2)$ -dimensional ambient manifold  $\mathcal{M}$  is reduced from  $\frac{(d+2)^2(d+1)}{2}$  to  $\frac{(d+1)^2d}{2}$ . The arbitrariness remaining in a projectable compatible torsion tensor is therefore the same as in a torsional gravitational fieldstrength  $(\bar{F}, \bar{U})$  measured by an arbitrary field of observers on the  $(d+1)$ -dimensional Platonic screen  $\bar{\mathcal{M}}$  (*cf.* Section 4.1 of [5]). A schematic argument regarding the previous Proposition is given by the following diagram:

$$\begin{array}{ccccc}
(\mathcal{M}, \xi, \psi, \gamma) & \xrightarrow{[N, A]} & (\mathcal{M}, \xi, g) & \xrightarrow{[N, F, [U]]} & (\mathcal{M}, \xi, g, \nabla) \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma}) & \xrightarrow{[\bar{N}, [A]]} & (\bar{\mathcal{M}}, \bar{\psi}, [\bar{g}]) & \xrightarrow{[\bar{N}, \bar{F}, \bar{U}]} & (\bar{\mathcal{M}}, \bar{\psi}, [\bar{g}], \bar{\nabla})
\end{array} \tag{3.51}$$

The logic underlying the left half of the diagram is the same as in the torsionfree case. Now, the right upper row relies on the following Lemma:

**Lemma 3.23.** *Let  $(\mathcal{M}, \xi, g)$  be an invariant fibered Lorentzian structure. The space of projectable connections compatible with  $(\mathcal{M}, \xi, g)$  forms an affine space canonically isomorphic to the affine space*

$$\frac{FLO_{\text{inv}}(\mathcal{M}, \psi, g) \times \left( \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi) \otimes \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi \otimes (\text{Ker } \psi / \text{Span } \xi)) \right)}{\Gamma_{\text{inv}}(\text{Ker } \psi / \text{Span } \xi)}. \tag{3.52}$$

**Proof:**

Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be the projectable ambient Leibnizian structure induced by  $(\mathcal{M}, \xi, g)$ . As showed in Proposition 3.14, the affine space  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  of ambient Galilean connections with projectable torsion compatible with  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  is canonically isomorphic to the affine space  $\mathcal{F}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  of invariant tor-

sional ambient gravitational fieldstrengths. Now, using the Lagrangian metric  $g \in \mathcal{L}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  allows to express the affine space of invariant torsional gravitational fieldstrengths as:

$$\mathcal{F}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma) \cong \frac{FLO_{\text{inv}}(\mathcal{M}, \psi, g) \times \mathcal{W}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)}{\Gamma_{\text{inv}}(\text{Ker } \psi / \text{Span } \xi)}$$

with  $\mathcal{W}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma) \equiv \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi) \oplus \Gamma_{\text{inv}}(\otimes^2 \text{Ann } \xi) \oplus \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi \otimes (\text{Ker } \psi / \text{Span } \xi))$ . As embodied in Lemma 3.19, given a torsional ambient Galilean connection  $\nabla \in \mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$ , the compatibility condition with the metric  $g$  totally fixes the  $\Gamma_{\text{inv}}(\otimes^2 \text{Ann } \xi)$  part (represented by  $\Sigma$ ) thus reducing the affine space  $\mathcal{D}(\mathcal{M}, \xi, \psi, \gamma)$  to eq.(3.52).  $\square$

A concrete realisation of the isomorphism (3.52) can be given as follows:

We let  $(\mathcal{M}, \xi, g)$  be an invariant fibered Lorentzian structure,  $N \in FLO_{\text{inv}}(\mathcal{M}, \psi, g)$  be an invariant field of light-like observers and denote  $A \equiv g(N)$ . The space of projectable connections compatible with  $(\mathcal{M}, \xi, g)$  forms an affine space isomorphic to the affine space of projectable torsion tensors satisfying conditions 1-2 of Proposition 3.16. Let  $\nabla$  be a projectable connection compatible with  $(\mathcal{M}, \xi, g)$  and  $T$  be the associated torsion tensor. We now make the further assumption that  $\nabla$  is an ambient Galilean connection for the invariant ambient Leibnizian structure underlying  $(\mathcal{M}, \xi, g)$  and denote  $\overset{N,A}{\Theta}(\nabla) = (F, \Sigma, [U])$ . By definition,  $[U] \in \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi \otimes (\text{Ker } \psi / \text{Span } \xi))$  is expressed in terms of  $T$  as  $[U] \equiv [P^N(T)]$ . More interestingly, according to Lemma 3.19, the 2-form  $F \in \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi)$  can also be expressed solely in terms of  $T$  as  $F = dA - A(T)$ . Conversely, the first condition of Lemma 3.19 can be interpreted as fixing the “light-like part” of the ambient torsion in terms of  $F$ . Taking into account the first constraint of Proposition 3.16 on its “timelike part”, the ambient projectable torsion can then be decomposed as:

$$T = N \otimes \psi(T) + \xi \otimes A(T) + P^L(T) = N \otimes d\psi + \xi \otimes (dA - F) + P^A(U). \quad (3.53)$$

Whenever  $T$  vanishes, one recover the familiar conditions:

- $d\psi = 0$  *i.e.* the induced metric structure is Augustinian
- $F = dA$  *i.e.* the nonrelativistic connection is Newtonian

- $[U] = 0$  *i.e.* the spatial torsion vanishes.

Returning to diagram 3.51, we conclude the proof of Proposition 3.22 by noting that the right vertical bijective map relies on the straightforward isomorphism

$$\frac{FLO_{\text{inv}}(\mathcal{M}, \psi, g) \times \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi) \otimes \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi \otimes (\text{Ker } \psi / \text{Span } \xi))}{\Gamma_{\text{inv}}(\text{Ker } \psi / \text{Span } \xi)} \cong \bar{\mathcal{F}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$$

with  $\bar{\mathcal{F}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  the affine space of nonrelativistic gravitational fieldstrengths defined as (*cf.* Definition 4.8 in [5]):

$$\bar{\mathcal{F}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma}) \equiv \frac{FO(\bar{\mathcal{M}}, \bar{\psi}) \times (\Omega^2(\bar{\mathcal{M}}) \oplus \Gamma(\wedge^2 T^* \bar{\mathcal{M}} \otimes \text{Ker } \bar{\psi}))}{\Gamma(\text{Ker } \bar{\psi})}$$

The previous line of reasoning provides a very general procedure aiming to embed any Galilean manifold into a projectable Lorentzian manifold. Explicitly, given an ambient structure  $(\mathcal{M}, \xi)$  with projection map  $\pi : \mathcal{M} \rightarrow \bar{\mathcal{M}}$ , any Leibnizian structure  $\bar{\mathcal{L}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  on the Platonic screen  $\bar{\mathcal{M}}$  can be lifted up to an ambient Leibnizian structure  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  (*cf.* Proposition 2.15) with  $\psi \equiv \pi^* \bar{\psi}$  and  $\gamma \equiv \pi^* \bar{\gamma}$ . Furthermore, we let  $\bar{\nabla} \in \bar{\mathcal{D}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  be a Galilean connection compatible with  $\bar{\mathcal{L}}(\bar{\mathcal{M}}, \bar{\psi}, \bar{\gamma})$  characterised by the torsional gravitational fieldstrength  $[\bar{N}, \bar{F}, \bar{U}]$ . The procedure goes on as follows: one needs to pick out an invariant ambient gravitational potential  $[N, A] \in \mathcal{P}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  so that to define an invariant ambient Lagrangian metric  $g \in \mathcal{L}_{\text{inv}}(\mathcal{M}, \xi, \psi, \gamma)$  on  $\mathcal{M}$  (*cf.* Proposition 2.31)<sup>10</sup>. The gravitational fieldstrength  $[\bar{N}, \bar{F}, \bar{U}]$  characterising the Galilean connection  $\bar{\nabla}$  can now be lifted up to its ambient equivalent  $[N, F, [U]]$  with  $N \in FLO_{\text{inv}}(\mathcal{M}, \psi, g)$  the unique light-like lift of  $\bar{N}$ ,  $F \in \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi)$  defined as  $F \equiv \pi^* \bar{F}$  and where  $[U] \in \Gamma_{\text{inv}}(\wedge^2 \text{Ann } \xi \otimes (\text{Ker } \psi / \text{Span } \xi))$  reads  $[U] \equiv [\pi^* \bar{U}]$ . As noted previously, one can define a unique torsion tensor  $T$

$$T \equiv N \otimes d\psi + \xi \otimes (dA - F) + P^A(U) \quad (3.54)$$

where  $(N, F, [U])$  is an arbitrary representative of  $[N, F, [U]]$  and  $A \equiv g(N)$ . Note that the expression of  $T$  is independent of the choice of representative  $(N, F, [U]) \in$

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<sup>10</sup>If one is given a Lagrangian structure on  $\bar{\mathcal{M}}$ , the ambient Lagrangian metric on  $\mathcal{M}$  is obtained from the class of nonrelativistic Lagrangian metrics (*cf.* Proposition 2.32).

$[N, F, [U]]$ . By construction, this tensor  $T$  is projectable and compatible with the invariant fibered Lorentzian structure  $(\mathcal{M}, \xi, g)$  so that the unique connection  $\nabla$  compatible with  $g$  and admitting  $T$  as torsion tensor is projectable and admits  $\bar{\nabla}$  as projection.

The construction previously sketched is very general and apply to any Galilean manifold. In particular, torsionfree Galilean manifolds which are non-Newtonian (*i.e.* the gravitational fieldstrength  $\bar{F}$  measured by an arbitrary observer is not necessarily closed) can be embedded into a Bargmann-Eisenhart wave endowed with a compatible connection whose torsion reads:

$$T \equiv \xi \otimes (dA - F).$$

Another important particular subcase consists in nonrelativistic Galilean manifolds whose connection is the torsional special connection  $\bar{\Gamma}^{\bar{N}}$  associated to a field of observers  $\bar{N} \in FO(\bar{\mathcal{M}}, \bar{\psi})$  (*cf.* Definition 4.4 in [5] as well as [15, 10, 11, 14] where these manifolds were referred to as Torsional Newton-Cartan (TNC) geometries. We refer especially to the works [15, 11] where these connections were investigated in the context of null dimensional reduction). From eq. (3.54), we find that these special Galilean manifolds can be embedded inside projectable Lorentzian manifolds whose ambient torsion takes the form

$$T = N \otimes d\psi + \xi \otimes dA.$$

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# A Through the Looking Glass: a compendium on Carrollian manifolds

The present Appendix can be seen as a mirror image of the work [5] where the principal definitions and results regarding intrinsic nonrelativistic Galilean geometry are dualised to the Carrollian case.

## A.1 Carrollian structures

**Definition A.1** (Carrollian metric). *Let  $(\mathcal{M}, \xi)$  be an ambient structure. A Carrollian metric on  $(\mathcal{M}, \xi)$  is a positive semi-definite covariant metric  $\gamma \in \Gamma(\vee^2 T^* \mathcal{M})$  whose radical is spanned by the fundamental vector field  $\xi$ . Alternatively, a Carrollian metric can be defined as a field  $h \in \Gamma(\vee^2 (\text{Ann } \xi)^*)$  on  $(\mathcal{M}, \xi)$  of positive-definite contravariant symmetric bilinear forms acting on 1-forms annihilating the fundamental vector field  $\xi$ .*

**Definition A.2** (Carrollian structure [39]). *A Carrollian structure consists of a triplet  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$  composed by the following elements:*

- an ambient structure  $(\mathcal{M}, \xi)$
- a Carrollian metric  $\gamma$ .

A Carrollian structure such that  $\mathcal{L}_\xi \gamma = 0$  will be said invariant.

Let  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$  be a Carrollian structure. The space of Ehresmann connections  $EC(\mathcal{M}, \xi)$  (cf. Definition 2.3) possesses the structure of an affine space modelled on  $\Gamma(\text{Ann } \xi)$ . The action of  $\Gamma(\text{Ann } \xi)$  on  $EC(\mathcal{M}, \xi)$  as  $A \mapsto A + \alpha$  with  $A \in EC(\mathcal{M}, \xi)$  will be referred to as a *Carroll boost* parameterised by the 1-form  $\alpha \in \Gamma(\text{Ann } \xi)$ . A principal connection ( $\mathcal{L}_\xi A = 0$ ) will be referred to as a principal connection, the affine space of principal connections being denoted  $PC(\mathcal{M}, \xi)$ .

**Example A.3** (Carroll spacetime). The most simple example of a Carrollian structure is given by the Carroll spacetime  $\mathcal{M} \cong \mathbb{R}^{d+1}$  with coordinates  $(u, x^i)$  characterised by the following fundamental vector field and (flat) Carrollian metric:

$$\begin{cases} \xi = \frac{\partial}{\partial u} \\ \gamma = \delta_{ij} dx^i \vee dx^j \end{cases}$$

where  $i, j \in \{1, \dots, d\}$  and  $\delta_{ij}$  is the Kronecker delta. Equivalently, one may consider the following contravariant metric:  $h = \delta^{ij} \frac{\partial}{\partial x^i} \vee \frac{\partial}{\partial x^j}$ .

**Definition A.4** (Transverse cometric). *Let  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$  be a Carrollian structure and  $A \in EC(\mathcal{M}, \xi)$  an Ehresmann connection on  $\mathcal{M}$ . The transverse cometric  ${}^A h \in \Gamma(\vee^2 T\mathcal{M})$  is defined by its action on 1-forms  $\alpha, \beta \in \Omega^1(\mathcal{M})$  as*

$${}^A h(\alpha, \beta) = h(\bar{P}^A(\alpha), \bar{P}^A(\beta)). \quad (\text{A.55})$$

The right-hand side of eq.(A.55) is well-defined since the image of the projector  $\bar{P}^A$  lies in  $\Gamma(\text{Ann } \xi)$ . The transverse cometric  ${}^A h$  can be easily shown to satisfy the two relations:

$$\begin{cases} {}^A h^{\mu\nu} A_\nu = 0 \\ {}^A h^{\mu\lambda} \gamma_{\lambda\nu} = \delta_\nu^\mu - \xi^\mu A_\nu. \end{cases} \quad (\text{A.56})$$

In fact, given  $A \in EC(\mathcal{M}, \xi)$ , there is a unique contravariant metric  ${}^A h \in \Gamma(\vee^2 T\mathcal{M})$  satisfying the conditions (A.56).

**Definition A.5** (Carrollian basis). *Let  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$  be a Carrollian structure. A Carrollian basis of the dual tangent space  $T_x^* \mathcal{M}$  at a point  $x \in \mathcal{M}$  is an ordered basis  $C_x^* = \{A_x, \theta^1|_x, \dots, \theta^d|_x\}$  with  $A_x$  satisfying  $A_x(\xi_x) = 1$  and  $\{\theta^1|_x, \dots, \theta^d|_x\}$  a basis of  $\text{Ann } \xi_x$  which is orthonormal with respect to  $h_x$ .*

Explicitly, the basis  $C_x^* = \{A_x, \theta^1|_x, \dots, \theta^d|_x\}$  must satisfy the conditions:

1.  $A_x(\xi_x) = 1$
2.  $\theta^i|_x(\xi_x) = 0, \forall i \in \{1, \dots, d\}$
3.  $h_x(\theta^i|_x, \theta^j|_x) = \delta_{ij}, \forall i, j \in \{1, \dots, d\}$ .

A basis of  $T_x \mathcal{M}$  dual to  $C_x^* = \{A_x, \theta^i|_x\}$  is given by  $C_x = \{\xi_x, e_i|_x\}$ , where the  $d$  vectors  $e_i|_x$  satisfy the requirement:  $\theta^i|_x(e_j|_x) = \delta_j^i$ . For instance, in the above case of the Carroll spacetime (Example A.3):  $A = du$  and  $\theta^i = dx^i$  define a Carrollian basis at any point.

**Proposition A.6.** *At each point  $x \in \mathcal{M}$ , the set of automorphisms of  $T_x^* \mathcal{M}$  mapping each Carrollian basis into another one forms a group isomorphic to the homogeneous Carroll group.*

Explicitly, the automorphisms preserving the collection of Carrollian bases can be represented as the following matrices

$$T = \begin{pmatrix} 1 & \mathbf{b}^T \\ 0 & \mathbf{R}^T \end{pmatrix} \quad (\text{A.57})$$

with  $\mathbf{b} \in \mathbb{R}^d$  and  $\mathbf{R} \in O(d)$ . This set of matrices forms a subgroup of  $GL(d+1, \mathbb{R})$  isomorphic to the homogeneous Carroll group  $\text{Carr}_0(d+1)$ . The homogeneous Carroll group therefore acts regularly on the space of Carrollian bases via the group action:

$$\{A, \theta^i\} \mapsto \{A + \mathbf{b}_i^T \theta^i, \mathbf{R}_{ij}^{T^i} \theta^j\}. \quad (\text{A.58})$$

Given  $C^* = \{A, \theta^i\}$ , one can define a dual basis  $C = \{\xi, e_i\}$ , with  $\theta^i(e_j) = \delta_j^i$  on which the homogeneous Carroll group acts transitively as:

$$\{\xi, e_i\} \mapsto \{\xi, \mathbf{R}_i^j (e_j - \mathbf{b}_j^T \xi)\}. \quad (\text{A.59})$$

**Definition A.7** (Carrollian manifold). *A Carrollian structure supplemented with a Koszul connection compatible with the fundamental vector field and Carrollian metric is called a Carrollian manifold. The Koszul connection is then referred to as a Carrollian connection.*

If we let  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$  be a Carrollian structure, the compatibility conditions explicitly read:

1.  $\nabla \xi = 0$
2.  $\nabla \gamma = 0$ .

**Proposition A.8.** *Let  $\mathcal{C}(\mathcal{M}, \xi, \gamma, \nabla)$  be a Carrollian manifold and denote  $T \in \Gamma(\wedge^2 T^* \mathcal{M} \otimes T \mathcal{M})$  the torsion tensor associated to the Carrollian connection  $\nabla$ .*



Galilei		Carroll	
Absolute clock $\psi \in \Omega^1(\mathcal{M})$		Fundamental vector field $\xi \in \Omega^1(\mathcal{M})$	
Collection of absolute rulers	$h \in \Gamma(\vee^2 T\mathcal{M})$ $\text{Rad } h \cong \text{Span } \psi$ <hr/> $\gamma \in \Gamma(\vee^2(\text{Ker } \psi)^*)$	Carrollian metric	$\gamma \in \Gamma(\vee^2 T^*\mathcal{M})$ $\text{Rad } \gamma \cong \text{Span } \xi$ <hr/> $h \in \Gamma(\vee^2(\text{Ann } \xi)^*)$
	Field of observers $N \in FO(\mathcal{M}, \psi)$		Ehresmann connection $A \in EC(\mathcal{M}, \xi)$
Milne boosts	$N \mapsto N + V$ with $V \in \Gamma(\text{Ker } \psi)$	Carroll boosts	$A \mapsto A + \alpha$ with $\alpha \in \Gamma(\text{Ann } \xi)$
<div>Transverse metric<hr/><math>\overset{N}{\gamma} \in \Gamma(\vee^2 T^*\mathcal{M})</math> <math>\overset{N}{\gamma}(X, Y) \equiv \gamma(P^N(X), P^N(Y))</math> for all <math>X, Y \in \Gamma(T\mathcal{M})</math></div>		<div>Transverse cometric<hr/><math>\overset{A}{h} \in \Gamma(\vee^2 T\mathcal{M})</math> <math>\overset{A}{h}(\alpha, \beta) \equiv h(\bar{P}^A(\alpha), \bar{P}^A(\beta))</math> for all <math>\alpha, \beta \in \Omega^1(\mathcal{M})</math></div>	
Torsion constraint			
$\psi(T(X, Y)) = d\psi(X, Y)$ for all $X, Y \in \Gamma(T\mathcal{M})$		$\mathcal{L}_\xi \gamma(V, W) = \gamma(V, T(\xi, W)) + \gamma(W, T(\xi, V))$ for all $V, W \in \Gamma(T\mathcal{M})$	

Table 1: Duality between Galilei and Carroll structures

The following relation holds:

$$\mathcal{L}_\xi \gamma(V, W) = \gamma(V, T(\xi, W)) + \gamma(W, T(\xi, V)) \text{ for all } V, W \in \Gamma(T\mathcal{M}).$$

In particular, one can conclude from the previous relation that only invariant Carrollian structures admit torsionfree compatible connections. This is just a particular instance of the standard requirement that the radical of a degenerate metric preserved by a torsionfree connection must be a Killing distribution (*cf. e.g.* Theorem 5.1 in [54]). The following table sums up the previously discussed Carrollian structures mirroring them with their Galilei duals:

## A.2 Torsionfree Carrollian connections

Let  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$  be an invariant Carrollian structure. The space of torsionfree connections compatible with  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$  will be denoted  $\mathcal{D}(\mathcal{M}, \xi, \gamma)$ .

**Proposition A.9.** *The space  $\mathcal{D}(\mathcal{M}, \xi, \gamma)$  of torsionfree Carrollian connections possesses the structure of an affine space modelled on the vector space*

$$\mathcal{V}(\mathcal{M}, \xi, \gamma) \equiv \left\{ S \in \Gamma(\vee^2 T^* \mathcal{M} \otimes T\mathcal{M}) \text{ satisfying conditions a)-b) } \right\}$$

where:

$$a) \quad S(X, \xi) = 0 \text{ for all } X \in \Gamma(T\mathcal{M})$$

$$b) \quad \gamma(S(X, Y), Z) + \gamma(S(X, Z), Y) = 0 \text{ for all } X, Y, Z \in \Gamma(T\mathcal{M}).$$

Note that condition b) as well as the symmetry of  $S$  ensures that  $\gamma(S(X, Y), Z) = 0$  for all  $X, Y, Z \in \Gamma(T\mathcal{M})$ . In other words,  $S(X, Y) \in \text{Span } \xi$  for all  $X, Y \in \Gamma(T\mathcal{M})$ .

**Lemma A.10.** *The vector space  $\mathcal{V}(\mathcal{M}, \xi, \gamma)$  is isomorphic to the vector space  $\Gamma(\vee^2 \text{Ann } \xi)$ .*

Explicitly, given a principal connection  $A \in EC(\mathcal{M}, \xi)$ , one can construct the following canonical isomorphism:

$$\varphi : \mathcal{V}(\mathcal{M}, \xi, \gamma) \rightarrow \Gamma(\vee^2 \text{Ann } \xi) : S_{\mu\nu}^\lambda \mapsto \Sigma_{\mu\nu} = A_\lambda S_{\mu\nu}^\lambda \quad (\text{A.60})$$

whose inverse takes the form

$$\varphi^{-1} : \Gamma(\vee^2 \text{Ann } \xi) \rightarrow \mathcal{V}(\mathcal{M}, \xi, \gamma) : \Sigma_{\mu\nu} \mapsto S_{\mu\nu}^\lambda = \xi^\lambda \Sigma_{\mu\nu}. \quad (\text{A.61})$$

We now construct the following map:

$$\Theta : PC(\mathcal{M}, \xi) \times \mathcal{D}(\mathcal{M}, \xi, \gamma) \rightarrow \Gamma(\vee^2 \text{Ann } \xi) : (A, \Gamma) \mapsto \overset{A}{\Sigma}_{\mu\nu} = -\nabla_{(\mu} A_{\nu)}.$$

For all principal connection  $A \in PC(\mathcal{M}, \xi)$ , the map  $\overset{A}{\Theta} \equiv \Theta(A, \cdot) : \mathcal{D}(\mathcal{M}, \xi, \gamma) \rightarrow \Gamma(\vee^2 \text{Ann } \xi)$  can be shown to be an affine map modelled on the linear map  $\varphi$  i.e.  $\overset{A}{\Theta}(\Gamma') - \overset{A}{\Theta}(\Gamma) = \varphi(\Gamma' - \Gamma)$ . Note that the fact that  $A$  is a principal connection ensures that  $\overset{A}{\Sigma} \in \Gamma(\vee^2 \text{Ann } \xi)$ .

The fact that  $\overset{A}{\Theta}$  is an affine map modelled on the isomorphism of vector spaces  $\varphi$  ensures that the kernel of  $\overset{A}{\Theta}$  is spanned by a unique  $\overset{A}{\Gamma} \in \mathcal{D}(\mathcal{M}, \xi, \gamma)$ . We call  $\overset{A}{\Gamma}$  the *torsionfree special Carrollian connection* associated to the principal connection  $A$ . An explicit expression of its coefficients is given by:

$$\overset{A}{\Gamma}_{\mu\nu}^{\lambda} = \xi^{\lambda} \partial_{(\mu} A_{\nu)} + \frac{1}{2} \overset{A}{h}^{\lambda\rho} [\partial_{\mu} \gamma_{\rho\nu} + \partial_{\nu} \gamma_{\rho\mu} - \partial_{\rho} \gamma_{\mu\nu}].$$

Given a principal connection  $A$ , the associated torsionfree special Carrollian connection can thus be used in order to represent any torsionfree Carrollian connection  $\Gamma \in \mathcal{D}(\mathcal{M}, \xi, \gamma)$  as  $\Gamma = \overset{A}{\Gamma} + \varphi^{-1} \left( \overset{A}{\Theta}(\Gamma) \right)$ . Explicitly, the components of  $\Gamma$  can be written as<sup>11</sup>:

$$\Gamma_{\mu\nu}^{\lambda} = \xi^{\lambda} \partial_{(\mu} A_{\nu)} + \frac{1}{2} \overset{A}{h}^{\lambda\rho} [\partial_{\mu} \gamma_{\rho\nu} + \partial_{\nu} \gamma_{\rho\mu} - \partial_{\rho} \gamma_{\mu\nu}] + \xi^{\lambda} \overset{A}{\Sigma}_{\mu\nu}.$$

This is the most general expression of a torsionfree Carrollian connection compatible with the Carrollian structure  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$ , the arbitrariness being encoded in the twice-covariant symmetric tensor  $\overset{A}{\Sigma} \in \Gamma(\vee^2 \text{Ann } \xi)$  (*cf.* [42] for a conjecture of this fact in the context of gauging procedures). The superscript acts again as a reminder of the fact that  $\overset{A}{\Theta}$  is not canonical. Under a Carroll boost parameterised by the invariant 1-form  $\alpha \in \Gamma_{\text{inv}}(\text{Ann } \xi)$ , the map  $\overset{A}{\Theta} : \mathcal{D}(\mathcal{M}, \xi, \gamma) \rightarrow \Gamma(\vee^2 \text{Ann } \xi)$  varies according to

$$\overset{A'}{\Theta}(\Gamma) = \overset{A}{\Theta}(\Gamma) - \frac{1}{2} \mathcal{L}_{\alpha^b} \gamma$$

for all  $\Gamma \in \mathcal{D}(\mathcal{M}, \xi, \gamma)$ , where  $\alpha^b \equiv \overset{A}{h}(\alpha) \in \Gamma_{\text{inv}}(T\mathcal{M})$ . This expression can then be

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<sup>11</sup>If one considers a generic Ehresmann connection  $A \in EC(\mathcal{M}, \xi)$ , the map  $\overset{A}{\Theta}$  becomes  $\overset{A}{\Theta} : \Gamma \mapsto \overset{A}{\Sigma}_{\mu\nu} = -\nabla_{(\mu} A_{\nu)} + A_{(\mu} \mathcal{L}_{\xi} A_{\nu)}$  while the generic torsionfree Carrollian connection reads:

$$\Gamma_{\mu\nu}^{\lambda} = \xi^{\lambda} \partial_{(\mu} A_{\nu)} + \frac{1}{2} \overset{A}{h}^{\lambda\rho} [\partial_{\mu} \gamma_{\rho\nu} + \partial_{\nu} \gamma_{\rho\mu} - \partial_{\rho} \gamma_{\mu\nu}] + \xi^{\lambda} \overset{A}{\Sigma}_{\mu\nu} - \xi^{\lambda} A_{(\mu} \mathcal{L}_{\xi} A_{\nu)}.$$

used to define a  $\Gamma_{\text{inv}}(\text{Ann } \xi)$ -action on the space  $PC(\mathcal{M}, \xi) \times \Gamma(\vee^2 \text{Ann } \xi)$  as:

$$\begin{aligned} & \left( PC(\mathcal{M}, \xi) \times \Gamma(\vee^2 \text{Ann } \xi) \right) \times \Gamma_{\text{inv}}(\text{Ann } \xi) \rightarrow PC(\mathcal{M}, \xi) \times \Gamma(\vee^2 \text{Ann } \xi) \\ & \left( (A, \Sigma), \alpha \right) \mapsto \left( A' = A + \alpha, \Sigma' = \Sigma - \frac{1}{2} \mathcal{L}_{\alpha^\flat} \gamma \right). \end{aligned} \quad (\text{A.62})$$

**Proposition A.11.** *The space of torsionfree Carrollian connections compatible with a given invariant Carrollian structure possesses the structure of an affine space canonically isomorphic to the following affine space of orbits:*

$$\frac{PC(\mathcal{M}, \xi) \times \Gamma(\vee^2 \text{Ann } \xi)}{\Gamma_{\text{inv}}(\text{Ann } \xi)} \cong \mathcal{D}(\mathcal{M}, \xi, \gamma).$$

### A.3 Torsional Carrollian connections

We let  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$  be a Carrollian structure and denote  $\mathcal{D}(\mathcal{M}, \xi, \gamma)$  the space of compatible Carrollian connections.

**Proposition A.12.** *The space  $\mathcal{D}(\mathcal{M}, \xi, \gamma)$  of compatible Carrollian connections possesses the structure of an affine space modelled on the vector space  $\mathcal{V}(\mathcal{M}, \xi, \gamma)$  defined as:*

$$\mathcal{V}(\mathcal{M}, \xi, \gamma) \equiv \left\{ S \in \Gamma(T^* \mathcal{M} \otimes T^* \mathcal{M} \otimes T \mathcal{M}) \text{ satisfying conditions a)-b)} \right\}$$

where

$$a) \quad S(X, \xi) = 0 \text{ for all } X \in \Gamma(T \mathcal{M})$$

$$b) \quad \gamma(S(X, Y), Z) + \gamma(S(X, Z), Y) = 0 \text{ for all } X, Y, Z \in \Gamma(T \mathcal{M}).$$

**Lemma A.13.** *The vector space  $\mathcal{V}(\mathcal{M}, \xi, \gamma)$  is isomorphic to the vector space  $\Gamma(T^* \mathcal{M} \otimes \text{Ann } \xi) \oplus \mathcal{U}(\mathcal{M}, \xi, \gamma)$  where*

$$\mathcal{U}(\mathcal{M}, \xi, \gamma) \equiv \left\{ U \in \Gamma(\text{Ann } \xi \otimes \wedge^2 T^* \mathcal{M}) \text{ such that } U(X|Y, \xi) + U(Y|X, \xi) = 0 \right\}.$$

Explicitly, given an Ehresmann connection  $A \in EC(\mathcal{M}, \xi)$ , one can construct the

following non-canonical isomorphism:

$$\begin{aligned} \overset{A}{\varphi} &: \mathcal{V}(\mathcal{M}, \xi, \gamma) \rightarrow \Gamma(T^*\mathcal{M} \otimes \text{Ann } \xi) \oplus \mathcal{U}(\mathcal{M}, \xi, \gamma) \\ &: S_{\mu\nu}^\lambda \mapsto \left( \Sigma_{\mu\nu} = A_\lambda S_{\mu\nu}^\lambda, U_{\lambda|\mu\nu} = \gamma_{\lambda\rho} S_{[\mu\nu]}^\rho \right) \end{aligned}$$

whose inverse takes the form

$$\begin{aligned} \overset{A}{\varphi}^{-1} &: \Gamma(T^*\mathcal{M} \otimes \text{Ann } \xi) \oplus \mathcal{U}(\mathcal{M}, \xi, \gamma) \rightarrow \mathcal{V}(\mathcal{M}, \xi, \gamma) \\ &: (\Sigma_{\mu\nu}, U_{\lambda|\mu\nu}) \mapsto S_{\mu\nu}^\lambda = \xi^\lambda \Sigma_{\mu\nu} - \overset{A}{h}^{\lambda\rho} (U_{\mu|\nu\rho} + U_{\nu|\mu\rho} - U_{\rho|\mu\nu}). \end{aligned}$$

We now construct the following map:

$$\begin{aligned} \Theta &: EC(\mathcal{M}, \xi) \times \mathcal{D}(\mathcal{M}, \xi, \gamma) \rightarrow \Gamma(T^*\mathcal{M} \otimes \text{Ann } \xi) \oplus \mathcal{U}(\mathcal{M}, \xi, \gamma) \\ &: (A, \Gamma) \mapsto \left( \Sigma_{\mu\nu} = -\nabla_\mu A_\nu, U_{\lambda|\mu\nu} = \gamma_{\lambda\rho} \Gamma_{[\mu\nu]}^\rho - \frac{1}{2} A_{[\mu} \mathcal{L}_\xi \gamma_{\nu]\lambda} \right). \end{aligned}$$

For all Ehresmann connection  $A \in EC(\mathcal{M}, \xi)$ , the map  $\overset{A}{\Theta} \equiv \Theta(A, \cdot) : \mathcal{D}(\mathcal{M}, \xi, \gamma) \rightarrow \Gamma(T^*\mathcal{M} \otimes \text{Ann } \xi) \oplus \mathcal{U}(\mathcal{M}, \xi, \gamma)$  can be shown to be an affine map modelled on the linear map  $\overset{A}{\varphi}$ . This fact ensures that  $\text{Ker } \overset{A}{\Theta}$  is spanned by a unique  $\overset{A}{\Gamma} \in \mathcal{D}(\mathcal{M}, \xi, \gamma)$ , called the *torsional special Carrollian connection* associated to  $A$ , whose coefficients read:

$$\overset{A}{\Gamma}_{\mu\nu}^\lambda = \xi^\lambda \partial_\mu A_\nu + \frac{1}{2} \overset{A}{h}^{\lambda\rho} [\partial_\mu \gamma_{\rho\nu} + \partial_\nu \gamma_{\rho\mu} - \partial_\rho \gamma_{\mu\nu}] - \frac{1}{2} \overset{A}{h}^{\lambda\rho} A_\nu \mathcal{L}_\xi \gamma_{\mu\rho}.$$

Given an Ehresmann connection  $A \in EC(\mathcal{M}, \xi)$ , the associated torsional special Carrollian connection can be chosen as origin of  $\mathcal{D}(\mathcal{M}, \xi, \gamma)$  and can thus be used in order to represent any torsional Carrollian connection: the components of the most general torsional Carrollian connection compatible with  $\mathcal{C}(\mathcal{M}, \xi, \gamma)$  thus read:

$$\Gamma_{\mu\nu}^\lambda = \overset{A}{\Gamma}_{\mu\nu}^\lambda + \xi^\lambda \Sigma_{\mu\nu} - \overset{A}{h}^{\lambda\rho} (U_{\mu|\nu\rho} + U_{\nu|\mu\rho} - U_{\rho|\mu\nu}).$$

The transformation law of the map  $\overset{A}{\Theta}$  under a Carroll boost induces an action of

$\Gamma(\text{Ann } \xi)$  on  $EC(\mathcal{M}, \xi) \times \left( \Gamma(T^*\mathcal{M} \otimes \text{Ann } \xi) \oplus \mathcal{U}(\mathcal{M}, \xi, \gamma) \right)$  as:

$$\begin{cases} A' = A + \alpha \\ \Sigma'_{\mu\nu} = \Sigma_{\mu\nu} - \partial_{[\mu}\alpha_{\nu]} - \frac{1}{2}\mathcal{L}_{\alpha^\flat}\gamma_{\mu\nu} + \overset{A}{h}{}^{\lambda\rho}\alpha_\lambda A_\nu \mathcal{L}_\xi \gamma_{\mu\rho} + \overset{A}{h}{}^{\lambda\rho}\alpha_\lambda (U_{\mu|\nu\rho} + U_{\nu|\mu\rho} - U_{\rho|\mu\nu}) \\ U'_{\lambda|\mu\nu} = U_{\lambda|\mu\nu} - \frac{1}{2}\alpha_{[\mu}\mathcal{L}_\xi \gamma_{\nu]\lambda} \end{cases}.$$

**Proposition A.14.** *The space of torsional Carrollian connections compatible with a given invariant Carrollian structure possesses the structure of an affine space canonically isomorphic to the affine space of orbits*

$$\frac{EC(\mathcal{M}, \xi) \times \left( \Gamma(T^*\mathcal{M} \otimes \text{Ann } \xi) \oplus \mathcal{U}(\mathcal{M}, \xi, \gamma) \right)}{\Gamma(\text{Ann } \xi)} \cong \mathcal{D}(\mathcal{M}, \xi, \gamma).$$

## B Technical proofs

### Proof of Proposition 2.13

We only prove the implication  $\gamma \Rightarrow h$ , the converse statement can be obtained by similar means. Let  $\gamma \in \Gamma(\vee^2(\text{Ker } \psi)^*)$  be a field of positive semi-definite covariant symmetric bilinear forms on  $\mathcal{M}$  satisfying  $\text{Rad } \gamma \cong \text{Span } \xi$  and  $\{\xi, e_1, \dots, e_d\}$  be a basis of  $\Gamma(\text{Ker } \psi)$  satisfying

$$\gamma(e_i, e_j) = \delta_{ij}, \quad \forall i, j \in \{1, \dots, d\}. \quad (\text{B.63})$$

Such a basis will be referred to as an orthogonal basis of  $\Gamma(\text{Ker } \psi)$ . Note that the set of endomorphisms of  $\Gamma(\text{Ker } \psi)$  mapping each orthogonal basis into another one forms a group isomorphic to the  $d$ -dimensional Euclidean group which acts regularly on the space of orthogonal bases via the group action:

$$\{\xi, e_i\} \mapsto \{\xi, \mathbf{b}_i \xi + \mathbf{R}_i^j e_j\} \quad (\text{B.64})$$

with  $\mathbf{b} \in \mathbb{R}^d$  and  $\mathbf{R} \in O(d)$ .

Let us define  $h \in \Gamma(\vee^2(\text{Ann } \xi)^*)$  by its action on a pair of 1-forms  $\alpha, \beta \in \Gamma(\text{Ann } \xi)$

as:

$$h(\alpha, \beta) = \delta^{ij} \alpha(e_i) \beta(e_j). \quad (\text{B.65})$$

From the group action (B.64), one concludes that  $h$  is invariant under a change of orthogonal basis. It can then be checked that  $\text{Rad } h \cong \text{Span } \psi$ .

### Proof of Proposition 2.31

The proof will rest on three Lemmas. We start by establishing that the space of ambient Lagrangian metrics possesses a structure of affine space:

**Lemma B.1.** *Let  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  be an ambient Leibnizian structure. The corresponding space  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  of ambient Lagrangian metrics possesses the structure of an affine space modelled on  $\Gamma(\text{Ann } \xi)$ .*

#### Proof:

We start by showing that  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  is an affine space modelled on the vector space

$$\mathcal{V}(\mathcal{M}, \xi, \psi) \equiv \{ \tilde{g} \in \Gamma(\vee^2 T^* \mathcal{M}) \mid \tilde{g}(\xi) = 0 \text{ and } \tilde{g}(V, W) = 0 \text{ for all } V, W \in \Gamma(\text{Ker } \psi) \}$$

by displaying the following subtraction map  $\tilde{g} = g' - g$  which can be shown to satisfy Weyl's axioms (*cf.* e.g. Appendix B of [5] for a reminder on affine spaces). Furthermore, we note that for all  $N \in FO(\mathcal{M}, \psi)$ , any element  $\tilde{g} \in \mathcal{V}(\mathcal{M}, \xi, \psi)$  can be written as

$$\tilde{g} = [2\tilde{g}(N) \vee \psi - \tilde{g}(N, N) \psi] \vee \psi. \quad (\text{B.66})$$

We now construct the canonical isomorphism:

$$\begin{aligned} \varphi &: \mathcal{V}(\mathcal{M}, \xi, \psi) \rightarrow \Gamma(\text{Ann } \xi) \\ &: \tilde{g} \mapsto \alpha \equiv \tilde{g}(N) - \frac{1}{2} \tilde{g}(N, N) \psi \text{ with } N \in FO(\mathcal{M}, \psi) \end{aligned}$$

together with its inverse

$$\begin{aligned}\varphi^{-1} &: \Gamma(\text{Ann } \xi) \rightarrow \mathcal{V}(\mathcal{M}, \xi, \psi) \\ &: \alpha \mapsto \tilde{g} \equiv 2\alpha \vee \psi.\end{aligned}$$

We note that the isomorphism  $\varphi$  is canonical *i.e.* does not depend on the choice of field of observers  $N$  as can be shown making use of equality B.66.  $\square$

We now introduce the map

$$\begin{aligned}\overset{N,A}{\Theta} &: \mathcal{L}(\mathcal{M}, \xi, \psi, \gamma) \rightarrow \Gamma(\text{Ann } \xi) \\ &: g \mapsto g(N) - \frac{1}{2}g(N, N)\psi - A\end{aligned}$$

which can be checked to be an affine map modelled on  $\varphi$  *i.e.*  $\overset{N,A}{\Theta}(g') - \overset{N,A}{\Theta}(g) = \varphi(g' - g)$  for all  $g', g \in \mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$ . According to Proposition B.4 of [5], for all  $(N, A) \in LP(\mathcal{M}, \xi, \psi)$ , the map  $\overset{N,A}{\Theta}$  endows the space of ambient Lagrangian metrics  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  with a structure of vector space whose origin  $\overset{N,A}{g} \in \text{Ker } \overset{N,A}{\Theta}$  can be shown to be the metric  $\overset{N,A}{g} \equiv \overset{N}{\gamma} + 2\psi \vee A$ . The map  $\overset{N,A}{\Theta}$  is then an isomorphism of vector spaces whose inverse is given by  $\overset{N,A}{\Theta}^{-1} : \Gamma(\text{Ann } \xi) \rightarrow \mathcal{L}(\mathcal{M}, \xi, \psi, \gamma) : \omega \mapsto \overset{N}{\gamma} + 2\psi \vee (A + \omega)$ . Using Proposition B.4, one can define the action  $\sigma : (LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) \times \mathcal{H}(d) \rightarrow LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)$  as

$$\left( (N, A, \omega), ([V], [\alpha], a) \right) \mapsto (N', A', \omega')$$

with

$$\begin{cases} N' = N + P^A(V) + a\xi \\ A' = A + \bar{P}^N(\alpha) - (a + \alpha(V))\psi \\ \omega' = \omega - (\bar{P}^N(\alpha) - (a + \alpha(V))\psi) + \overset{N}{\gamma}(V) - \frac{1}{2}\gamma(V, V)\psi. \end{cases}$$

**Lemma B.2.** *The space  $(LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) / \mathcal{H}(d)$  of  $\mathcal{H}(d)$ -orbits in  $LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)$  is an affine space canonically isomorphic to the affine space of Lagrangian metrics  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$ .*



**Proof:**

According to Proposition B.4 of [5], the space  $(LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) / \mathcal{H}(d)$  possesses a structure of affine space modelled on  $\mathcal{V}(\mathcal{M}, \xi, \psi) \cong \Gamma(\text{Ann } \xi)$  with subtraction map:

$$([N, A, \omega'], [N, A, \omega]) \mapsto \omega' - \omega \quad (\text{B.67})$$

where we picked an arbitrary element  $(N, A) \in LP(\mathcal{M}, \xi, \psi)$  and where  $\omega$  (respectively  $\omega'$ ) is the unique element of  $\Gamma(\text{Ann } \xi)$  such that  $(N, A, \omega) \in [N, A, \omega]$  (respectively  $(N, A, \omega') \in [N, A, \omega']$ ). The existence and uniqueness of  $\omega \in \Gamma(\text{Ann } \xi)$  (resp.  $\omega' \in \Gamma(\text{Ann } \xi)$ ) are guaranteed by the fact that  $\mathcal{H}(d)$  acts regularly on  $LP(\mathcal{M}, \xi, \psi)$ . Finally, the difference  $\omega' - \omega$  is independent of the choice of  $(N, A) \in LP(\mathcal{M}, \xi, \psi)$ .

One can now display the explicit expression of the canonical isomorphism between the affine space  $(LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) / \mathcal{H}(d)$  and the affine space of Lagrangian metrics  $\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma)$  as:

$$\begin{aligned} \mathcal{L}(\mathcal{M}, \xi, \psi, \gamma) &\rightarrow (LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) / \mathcal{H}(d) \\ g &\mapsto \left[ N, A, g(N) - \frac{1}{2}g(N, N)\psi - A \right] \text{ for a given } (N, A) \in LP(\mathcal{M}, \xi, \psi) \end{aligned}$$

together with its inverse

$$\begin{aligned} (LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) / \mathcal{H}(d) &\rightarrow \mathcal{L}(\mathcal{M}, \xi, \psi, \gamma) \\ [N, A, \omega] &\mapsto \frac{N}{\gamma} + 2\psi \vee (A + \omega) \text{ for a given } (N, A, \omega) \in [N, A, \omega] \end{aligned}$$

which can be shown to be affine maps modelled on  $\varphi : \mathcal{V}(\mathcal{M}, \xi, \psi) \rightarrow \Gamma(\text{Ann } \xi)$  and  $\varphi^{-1} : \Gamma(\text{Ann } \xi) \rightarrow \mathcal{V}(\mathcal{M}, \xi, \psi)$ , respectively.  $\square$

**Lemma B.3.** *The affine space  $(LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) / \mathcal{H}(d)$  of  $\mathcal{H}(d)$ -orbits in  $LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)$  is canonically isomorphic to the affine space of ambient gravitational potentials  $\mathcal{P}(\mathcal{M}, \xi, \psi, \gamma)$ .*

**Proof:**

We start by noting that any  $\mathcal{H}(d)$ -orbit  $[N, A, \omega] \in (LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) / \mathcal{H}(d)$

contains a subspace of elements  $(N, A, \omega) \in LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)$  such that  $\omega$  vanishes. Explicitly, given any  $(N, A, \omega) \in [N, A, \omega]$ , one can cancel the 1-form  $\omega$  by acting via  $\sigma$  with  $([0], [\bar{P}^N(\omega)], -\omega(N)) \in \mathcal{H}(d)$  such that :

$$(N, A, \omega) \mapsto (N - \omega(N)\xi, A + \omega, 0). \quad (\text{B.68})$$

The previous expression motivates the introduction of the following map

$$\begin{aligned} \chi &: LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi) \rightarrow LP(\mathcal{M}, \xi, \psi) \\ &: (N, A, \omega) \mapsto (N - \omega(N)\xi, A + \omega) \end{aligned}$$

which can be shown to be a  $\mathcal{H}(d)$ -space isomorphism *i.e.*  $\chi$  satisfies

$$\chi\left(\sigma\left((N, A, \omega), ([V], [\alpha], a)\right)\right) = \tau\left(\chi\left((N, A, \omega)\right), R\left([V], [\alpha], a\right)\right)$$

for all  $(N, A, \omega) \in LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)$  and  $([V], [\alpha], a) \in \mathcal{H}(d)$  while the map  $R : \mathcal{H}(d) \rightarrow \Gamma(\text{Ker } \psi / \text{Span } \xi)$  is defined as  $R : ([V], [\alpha], a) \mapsto [V]$  and  $\tau : (LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ker } \psi / \text{Span } \xi)) \rightarrow LP(\mathcal{M}, \xi, \psi)$  is the group action defined in eq.(2.27). This justifies that the following map is a well-defined isomorphism

$$\begin{aligned} (LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) / \mathcal{H}(d) &\rightarrow \mathcal{P}(\mathcal{M}, \xi, \psi, \gamma) \\ [N, A, \omega] &\mapsto [\chi(N, A, \omega)]. \end{aligned} \quad (\text{B.69})$$

The subtraction map of  $\mathcal{P}(\mathcal{M}, \xi, \psi, \gamma)$  can be derived from the one of  $(LP(\mathcal{M}, \xi, \psi) \times \Gamma(\text{Ann } \xi)) / \mathcal{H}(d)$  and reads:

$$S([N', A'], [N, A]) = A' - A - \frac{N}{\gamma}(N') + \frac{1}{2}\frac{N}{\gamma}(N', N')\psi.$$

This expression can be shown to be independent of the choice of representatives  $(N, A)$  and  $(N', A')$ . Finally, we display the explicit form of the affine isomorphism between the affine spaces of Lagrangian metrics and ambient

gravitational potentials:

$$\begin{aligned}\mathcal{L}(\mathcal{M}, \xi, \psi, \gamma) &\rightarrow \mathcal{P}(\mathcal{M}, \xi, \psi, \gamma) \\ g &\mapsto \left[ N - \frac{1}{2}g(N, N)\xi, g(N) - \frac{1}{2}g(N, N)\psi \right] \text{ for a given } N \in FO(\mathcal{M}, \psi)\end{aligned}$$

together with its inverse

$$\begin{aligned}\mathcal{P}(\mathcal{M}, \xi, \psi, \gamma) &\rightarrow \mathcal{L}(\mathcal{M}, \xi, \psi, \gamma) \\ [N, A] &\mapsto \frac{N}{\gamma} + 2\psi \vee A \text{ for a given } (N, A) \in [N, A]\end{aligned}$$

which can be shown to be affine maps modelled on  $\varphi : \mathcal{V}(\mathcal{M}, \xi, \psi) \rightarrow \Gamma(\text{Ann } \xi)$  and  $\varphi^{-1} : \Gamma(\text{Ann } \xi) \rightarrow \mathcal{V}(\mathcal{M}, \xi, \psi)$ , respectively.  $\square$

This concludes the proof of Proposition 2.31.

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